University of Alabama
Department of Physics and Astronomy

PH 105 LeClair Summer 2015

Week 3 Homework - Solutions

Problems for 9 June (due 10 June)

1. On a frictionless table, a mass m moving at speed v collides with another mass m initially at rest. The masses stick together. How much energy is converted to heat?

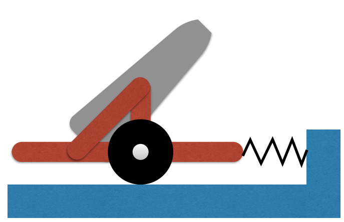
Solution: You can get the final velocity with the 1D inelastic collision equation (or just conserve momentum). The initial kinetic energy is $\frac{1}{2}mv^2$, the final kinetic energy is $\frac{1}{4}mv^2$, so $\frac{1}{4}mv^2$ is lost to heat.

2. A very light ping pong ball bounces elastically head-on off a very heavy bowling ball that is initially at rest. What is the fraction of the ping pong ball's initial kinetic energy that is transferred to the bowling ball?

Solution: Use the 1D elastic collision equation with $v_{2i}=0$ and let $m_2 \to \infty$ to represent the bowling ball. This gives $v_{1f} \approx v_{1i}$, the ping pong ball has essentially the same speed after the collision. That means its kinetic energy is approximately the same, and almost no energy is transferred to the bowling ball.

3. A large howitzer is rigidly attached to a carriage, which can move along horizontal rails but is connected to a sturdy wall by a large spring, initially unstretched and with force constant $k = 1.90 \times 10^4\,\mathrm{N/m}$, as shown below. The cannon fires a 200 kg projectile at a velocity of 125 m/s directed 45.0° above the horizontal.

If the mass of the cannon and its carriage is 4780 kg, find the maximum extension of the spring.



Solution: First, we want to find the recoil velocity of the cannon, from which we can use conservation of energy to get the maximal extension of the spring.

We can get the recoil velocity from conservation of momentum, but we have to be careful. The projectile's momentum has both x and y components, but the howitzer will only move in the $-x$ direction. We have to write down conservation of momentum my components. In this case we only need the x components.

$$\begin{aligned}
\vec{p}_i &= \vec{p}_f \\
p_{xi} &= p_{xf} \\
0 &= v_{\text{proj}} m_{\text{proj}} \cos 45^\circ + m_{\text{howitzer}} v_{\text{howitzer}} \\
v_{\text{howitzer}} &= -\left(\frac{m_{\text{proj}}}{m_{\text{howitzer}}}\right) v_{\text{proj}} \cos 45^\circ
\end{aligned}$$

Now that we have the recoil velocity of the cannon, we can use conservation of energy to relate the cannon's kinetic energy to the energy stored in the spring.

$$\begin{aligned}
\frac{1}{2} m_{\text{howitzer}} v_{\text{cannon}}^2 &= \frac{1}{2} k (\Delta x)^2 \\
\Delta x &= \sqrt{\frac{m_{\text{howitzer}}}{k}} v_{\text{howitzer}} \\
\Delta x &= \sqrt{\frac{m_{\text{howitzer}}}{k}} \left(\frac{m_{\text{proj}}}{m_{\text{howitzer}}}\right) v_{\text{proj}} \cos 45^\circ \\
\Delta x &= \left[\frac{m_{\text{proj}}}{\sqrt{k m_{\text{howitzer}}}}\right] v_{\text{proj}} \cos 45^\circ \\
\Delta x &= \left[\frac{200\,\text{kg}}{\sqrt{(1.90 \times 10^4\,\text{N/m})\,(4780\,\text{kg})}}\right] (125\,\text{m/s}) \left(\frac{\sqrt{2}}{2}\right) \approx 1.85\,\text{m}
\end{aligned}$$

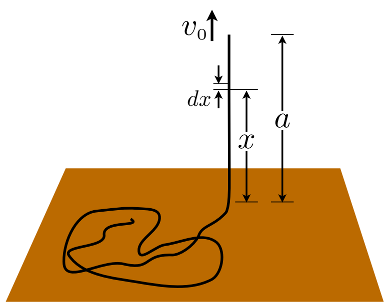
Problems for 10 June (due 11 June)

4. A uniformly dense rope of length b and mass per unit length λ is coiled on a smooth table. One end is lifted by hand with constant velocity v_o. Find the force of the rope on the hand when the rope is a distance a above the table ($b > a$).

Solution: **Find:** The force a rope exerts on a hand pulling it upward off of a table, as a function of position. The hand will have to pull against the rope's weight, but also against the changing momentum of the rope as more of it leaves the table.

Given: The length b and linear mass density λ, the constant velocity at which the rope is pulled.

Sketch: We want to know the total force between the hand and rope when a length a of the rope has been pulled off of the table at constant speed v_o.



Take a small segment of rope dx a height x off of the table, as shown in the sketch above, with the $+x$ direction being upward. This small segment has mass $dm = \lambda dx$, and was pulled off of the table at constant velocity v_o. Just before the segment was pulled off of the table, it was simply lying there with zero velocity and hence zero momentum. An instant later, it is moving away from the table at velocity v_o, which clearly implies a non-zero momentum. This means that during the time dt it took to pull the segment dx off of the table completely, its momentum changed from 0 to p_f. This time rate of change of momentum is a force.

Relevant equations: The main equation we will need is that force is the time rate of change of momentum:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Additionally, we need to know the weight of an arbitrary length of rope. Take a small section of rope of length a. Since the mass per unit length of the rope is λ, the mass of that segment must be λa, and its weight $\lambda g dx$.

Symbolic solution: Consider again our segment of rope dx. It has mass dm and velocity v_o just after it leaves the table, and zero velocity just before. The momentum change dp in pulling that segment of rope off of the table is

$$dp = v_o dm = v_o \lambda dx$$

If this segment took dp to pull off of the table, we can easily find the time rate of change of momentum by dividing by dt:

$$\frac{dp}{dt} = v_o \lambda \frac{dx}{dt} = v_o^2 \lambda$$

Here we used the fact that dx/dt is simply the velocity of the rope, which were are given as v_o. This is the impulse force that brings the string off the table, and which also acts on the hand pulling it off of the table. This impulse force is independent of how much rope is already off of the table, which makes sense: it only involves changing the momentum of an infinitesimal bit of rope at one instant, and does not depend on what the rest of the rope is doing. Since the bit of rope changes its velocity from zero to straight upward, the impulse that the hand feels must act in the downward direction by Newton's third law. That is, the force acting on the hand F_i must be equal and opposite of the impulse force acting on the rope, which is equal to the rope's time rate of change in momentum:

$$F_i = -\frac{dp}{dt} = -v_o^2 \lambda$$

In addition to the impulse, the hand must also support the weight of the string already off of the table. A length a of the rope must have mass λa, and therefore the hand must support a weight of $W = -\lambda g a$, also acting downward. The total force on the hand is this weight plus the impulse force:

$$F_{tot} = W + F_i = -\lambda g a - \lambda v_o^2 = -\lambda g a \left(1 + \frac{v_o^2}{ag}\right)$$

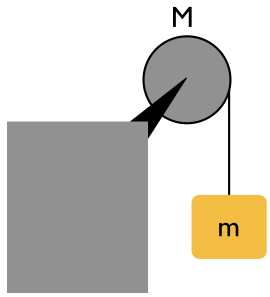
Numeric solution: Once again, there are no numbers given.

Double check: Dimensionally, our answer is correct. Checking each term in our force balance, noting that λ must have units of kilograms per meter

$$\lambda g a = \left[\text{kg}\,\text{m}^{-1}\right]\left[\text{m}\,\text{s}^2\right][\text{m}] = \left[\text{kg}\,\text{m/s}^2\right] = [\text{N}]$$
$$\lambda v_o^2 = \left[\text{kg}\,\text{m}^{-1}\right]\left[\text{m}^2\,\text{s}^2\right] = [\text{N}]$$

Our answer also makes sense qualitatively: both the impulse and weight force should get larger as λ increases (i.e., as the rope gets heavier). As the total length of rope above the table a increases, the weight should increase while the impulse force remains constant, which also makes sense. Finally, the impulse force should increase as the pulling speed v_o increases, while the weight should be unaffected.

5. A uniform disk with mass $M = 2.5\,\text{kg}$ and radius $R = 20\,\text{cm}$ is mounted on a fixed horizontal axle, as shown below. A block of mass $m = 1.2\,\text{kg}$ hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord. *Note: the moment of inertia of a disk about its center of mass is* $I = \frac{1}{2}MR^2$.



Solution: In order to get acceleration and angular acceleration, we'll need to use force and torque, respectively. Start with the pulley. The tension T in the rope pulls on the edge of the disk a distance R from the center of rotation at an angle of $\theta_{RT} = 90°$, which causes a torque τ. This torque must equal the disk's moment of inertia times the angular acceleration.

$$\tau = RT \sin\theta_{RT} = RT = I\alpha = \frac{1}{2}MR^2\alpha \tag{1}$$

$$\alpha = \frac{2T}{MR} \tag{2}$$

We can get the tension by considering the force balance for the hanging mass. We have the tension in the tope pulling up, the weight of the mass pulling down, and an overall acceleration downward. Thus

$$\sum F = T - mg = -ma \tag{3}$$

Noting that $a = R\alpha$, this gives $T = mg - MR\alpha$. Now we've got two equations for α, which we can combine.

$$\alpha = \frac{2T}{MR} = \frac{2}{MR}(mg - mR\alpha) = \frac{2mg}{MR} - \frac{2m}{M}\alpha \tag{4}$$

$$\frac{2mg}{MR} = \alpha\left(1 + \frac{2m}{M}\right) \tag{5}$$

$$\alpha = \frac{2mg}{R(M+2m)} \approx -24\,\mathrm{rad/s^2} \tag{6}$$

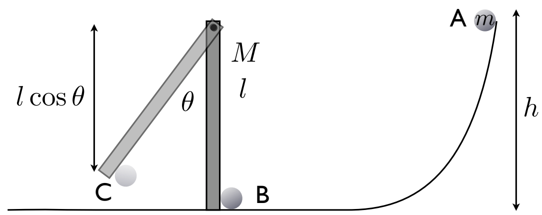
Given α, we can find a and T.

$$a = R\alpha = \frac{2mg}{M+2m} \approx -4.8\,\mathrm{m/s^2} \tag{7}$$

$$T = mg - MR\alpha = mg - \frac{2m^2g}{M+2m} = mg\left(1 - \frac{2m}{M+2m}\right) = g\left(\frac{mM}{M+2m}\right) \approx 6.0\,\mathrm{N} \tag{8}$$

Problems for 11 June (due 12 June)

6. In the figure below, a small block of mass m slides down a frictionless surface through height h and then sticks to a uniform rod of mass M and length L. The rod pivots about point O through angle θ before momentarily stopping. Find θ.



Solution: **Solution:** Referring to the sketch above, let A be the starting point, B the moment of collision between the ball and rod, and C the point when maximum height is reached by the rod + ball system. We approximate the ball as a point mass, since we are told it is small (and we anyway have no way of calculating its moment of inertia, since we do not have any geometrical details . . .).

The velocity v of the ball at point B can be found using conservation of mechanical energy. Let the floor be the height of zero gravitational potential energy.

$$\begin{aligned} K_A + U_A &= K_B + U_B \\ mgh &= \frac{1}{2}mv^2 \\ \implies v &= \sqrt{2gh} \end{aligned}$$

The collision is clearly inelastic, since the ball sticks to the rod. We could use conservation of linear momentum, but this would require breaking up the rod into infinitesimal discrete bits of mass and integrating over its length. Easier is to use conservation of *angular* momentum about the pivot point of the rod. Just before the collision, we have the ball moving at speed v a distance l. Let $\hat{\imath}$ be to the right, and $\hat{\jmath}$ upward (making $\hat{\mathbf{k}}$ into the page). The initial angular momentum is then

$$\vec{\mathbf{L}}_i = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = l\,\hat{\jmath} \times (-mv\,\hat{\imath}) = -mvl\,(\hat{\jmath} \times \hat{\imath}) = mvl\,\hat{\mathbf{k}} = ml\sqrt{2gh}\,\hat{\mathbf{k}}$$

After the collision, we have the rod and mass stuck together, rotating at angular velocity ω. Defining counterclockwise rotation to be positive as usual, the final angular momentum is thus

$$\vec{\mathbf{L}}_f = I\omega\,\hat{\mathbf{k}}$$

The total moment of inertia about the pivot point is that of the rod rotating plus that of the ball.

The rod rotates a distance $l/2$ from its center of mass, and again we approximate the ball as a point mass rotating at a distance l (since we told it is small).

$$I = I_{\text{rod}} + I_{\text{ball}} = I_{\text{rod, com}} + M\left(\frac{l}{2}\right)^2 + ml^2 = \frac{1}{12}Ml^2 + Ml^2 + ml^2 = \left(\frac{1}{3}M + m\right)l^2$$

Equating initial and final angular momentum, we can solve for the angular velocity after the collision:

$$L_f = I\omega = L_i = mvl = ml\sqrt{2gh}$$
$$\left(\frac{1}{3}M + m\right)l^2\omega = ml\sqrt{2gh}$$
$$\omega = \frac{m\sqrt{2gh}}{\left(\frac{1}{3}M + m\right)l}$$

At this point, we may use conservation of energy once again. When the system reaches its maximum angle θ at C, the center of mass of the rod + ball system will have moved up by an amount Δy_{cm}. The change in gravitational potential energy related to this change in center of mass height must be equal to the rotational kinetic energy just after the collision. Thus,

$$\frac{1}{2}I\omega^2 = \frac{\vec{\mathbf{L}} \cdot \vec{\mathbf{L}}}{2I} = \frac{L^2}{2I} = (m + M)\,g\Delta y_{cm}$$

Here we have noted that the rotational kinetic energy can be related to the angular momentum to save a bit of algebra. To proceed, we must find the difference in the center of mass height between points C and B. Let $y=0$ be the height of the floor. At point B,

$$y_{cm,B} = \frac{M\left(\frac{L}{2}\right) + m\,(0)}{m + M} = \left(\frac{l}{2}\right)\left(\frac{M}{m + M}\right)$$

At point C, the ball is now at a height $l - l\cos\theta$, while the center of mass of the rod (its midpoint) is now at $l - l\cos\theta + \frac{1}{2}l\cos\theta$. Thus,

$$y_{cm,C} = \frac{M\left(l - l\cos\theta + \frac{1}{2}l\cos\theta\right) + m\,(l - l\cos\theta)}{m + M} = \frac{Ml\left(1 - \frac{1}{2}\cos\theta\right) + ml\,(1 - \cos\theta)}{m + M}$$

The change in center of mass height can now be found:

$$\Delta y_{cm} = y_{cm,C} - y_{cm,B} = \frac{Ml\left(1-\frac{1}{2}\cos\theta\right) + ml\left(1-\cos\theta\right) - \frac{1}{2}Ml}{m+M}$$

$$= \frac{\frac{1}{2}Ml\left(1-\cos\theta\right) + ml\left(1-\cos\theta\right)}{m+M}$$

$$= \frac{l}{m+M}\left(1-\cos\theta\right)\left(m+\frac{1}{2}M\right)$$

Using our previous energy balance between B and C,

$$\frac{L^2}{2I} = (m+M)\,g\Delta y_{cm} = lg\left(1-\cos\theta\right)\left(m+\frac{1}{2}M\right)$$

We could also have found the change in potential energy a bit more easily by just separately considering the change energy due to the change in height of the center of mass of the rod and the ball separately and adding the two together. The putty changes height by $l{-}l\cos\theta$, while the rod's center of mass changes height by half that much.

$$\Delta U_{\text{ball}} = mgh_{\text{ball}} = mgl\left(1-\cos\theta\right) \tag{9}$$

$$\Delta U_{\text{rod}} = Mg_{\text{rod, cm}} = Mg\frac{l}{2}\left(1-\cos\theta\right) \tag{10}$$

$$\Delta U_{\text{tot}} = \Delta U_{\text{ball}} + \Delta U_{\text{rod}} = gl\left(1-\cos\theta\right)\left(m+\frac{1}{2}M\right) \tag{11}$$

Since the initial and final angular momenta are equal, we may substitute either L_f or L_i, the latter being the easiest option. This is not strictly *necessary* – we could use L_f or even just grind through $\frac{1}{2}I\omega^2$ and the result must be the same. However, using L_i here saves quite a bit of algebra in the end when we try to put θ in terms of only given quantities. Doing so, and solving for θ

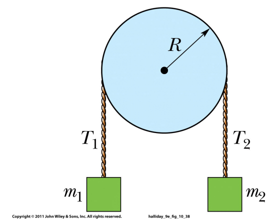
$$\frac{L_f^2}{2I} = \frac{L_i^2}{2I} = \frac{2l^2m^2gh}{2\left(\frac{1}{3}M+m\right)l^2} = lg\left(1-\cos\theta\right)\left(m+\frac{1}{2}M\right)$$

$$1-\cos\theta = \frac{m^2h}{l\left(\frac{1}{3}M+m\right)\left(\frac{1}{2}M+m\right)}$$

$$\theta = \cos^{-1}\left[1 - \frac{m^2h}{l\left(\frac{1}{3}M+m\right)\left(\frac{1}{2}M+m\right)}\right]$$

Note that for $m{=}0$, $\theta{=}0$, as we expect. On the other hand, for $M{=}0$ we have $\cos\theta{=}1{-}h/l{=}1/2$. This means that the particle is at a height $l-l\cos\theta{=}l/2{=}h$ at point C – exactly what we would expect if mechanical energy were conserved!

7. In the figure below, block 1 has mass m_1, block 2 has mass m_2 (with $m_2 > m_1$), and the pulley (a solid disc), which is mounted on a horizontal axle with negligible friction, has radius R and mass M. When released from rest, block 2 falls a distance d in t seconds without the cord slipping on the pulley. **(a)** What are the magnitude of the accelerations of the blocks? **(b)** What is T_1? **(c)** What is T_2? **(d)** What is the pulley's angular acceleration? The moment of inertia of a solid disc is $I = \frac{1}{2}MR^2$.



Solution: Give $m_2 > m_1$, we expect a clockwise rotation. Taking the positive y direction as upward, that makes the acceleration of mass 2 negative and that of mass 1 positive. We need to do two thing: first, balance the forces on the hanging masses, and two, analyze the torque on the disc.

With the sign conventions noted above, the forces are

$$T_2 - m_2 g = -m_2 a \tag{12}$$

$$T_1 - m_1 g = m_1 a \tag{13}$$

What we must be careful about now are the facts that the tension in each side of the rope is *not* just the weight of the hanging mass (this can't be true if the masses are accelerating, as the equations above indicate), and we should not assume that $T_1 = T_2$ when we have the pully's moment of inertia to consider. That means we have three unknowns (T_1, T_2, and a) but only two equations. Adding the torque analysis gets us the last equation we need.

$$\sum \tau = RT_2 - RT_1 = R(T_2 - T_1) = I\alpha \tag{14}$$

Noting that $\alpha = a/R$, one can solve the resulting equations for T_1, T_2, and a. The angular acceleration is also readily found. I'll assume you can work out the details:

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2 + \frac{1}{2}M} \right) g \tag{15}$$

$$\alpha = \left(\frac{m_2 - m_1}{m_1 + m_2 + \frac{1}{2}M} \right) \frac{g}{R} \tag{16}$$

$$T_1 = \left(\frac{2m_1 m_2 + \frac{1}{2}M m_1}{m_1 + m_2 + \frac{1}{2}M} \right) g \tag{17}$$

$$T_2 = \left(\frac{2m_1 m_2 + \frac{1}{2}M m_2}{m_1 + m_2 + \frac{1}{2}M} \right) g \tag{18}$$

How do we know this is plausible? We can set $I = 0$ to ignore the effect of the pulley, which reduces the system to the simple case of two masses on an ideal massless pulley that we've already studied. With $I = 0$:

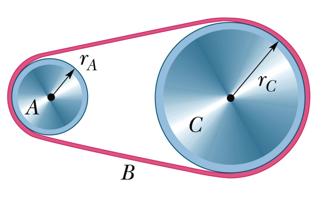
$$a = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) g \tag{19}$$

$$T_1 = \left(\frac{2m_1 m_2}{m_1 + m_2} \right) g \tag{20}$$

$$T_2 = \left(\frac{2m_1 m_2}{m_1 + m_2} \right) g \tag{21}$$

Now we see $T_1 = T_2$, and the tensions and acceleration are just what we found before.

8. A flywheel rotating freely on a shaft is suddenly coupled by means of a drive belt to a second flywheel sitting on a parallel shaft (see figure below). The initial angular velocity of the first flywheel is ω, that of the second is zero. The flywheels are uniform discs of masses M_a and M_c with radii R_a and R_c respectively. The moment of inertia of a solid disc is $I = \frac{1}{2}MR^2$. The drive belt is massless and the shafts are frictionless. **(a)** Calculate the final angular velocity of each flywheel. **(b)** Calculate the kinetic energy lost during the coupling process. *Hint: if the belt does not slip, the* **linear** *speeds of the two rims must be equal.*



Solution: If the belt doesn't slip, the linear velocity of the wheels must be the same at their outer rim when the final state is reached. That implies

$$v_a = v_c \tag{22}$$
$$R_a\omega_a = R_c\omega_c \tag{23}$$
$$\omega_c = \frac{R_a}{R_c}\omega_a \tag{24}$$

The sudden coupling of the second flywheel is basically a collision, and as is usually the case with collisions, conservation of energy is not a viable approach (how would you figure out how much energy the collision cost?). Conservation of momentum, or angular momentum when we have a rotation problem, is the way to go. Initially we have only the first flywheel rotating at ω, after the fact both are rotating. Conservation of angular momentum, combined with the relationship between ω_a and ω_c gives:

$$L_i = L_f \tag{25}$$
$$I_a\omega = \omega_a I_a + \omega_c I_c = \omega_a I_a + \frac{R_a}{R_c}\omega_a I_c \tag{26}$$
$$\Longrightarrow \qquad \omega_a = \frac{\omega}{1 + R_a I_c/R_c I_a} \tag{27}$$

Using the fact that the moments of inertia are $I = \frac{1}{2}MR^2$,

$$\omega_a = \frac{\omega}{1 + M_c R_c/M_a R_a} \tag{28}$$

The kinetic energy loss is straightforward to calculate, if messy.

$$K_f = \frac{1}{2}I_a\omega_a^2 + \frac{1}{2}I_c\omega_c^2 = \frac{1}{2}I_a\omega_a^2\left(1 + \frac{I_c R_a^2}{I_a R_c^2}\right) \tag{29}$$
$$K_i = \frac{1}{2}I_a\omega^2 \tag{30}$$

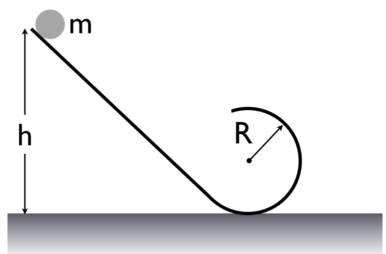
With a bit of algebra, you can work out the ratio

$$\frac{K_f}{K_i} = \frac{M_a R_a^2\left(M_a + M_c\right)}{\left(M_a R_a + M_c R_c\right)^2} \tag{31}$$

9. A solid sphere, a solid cylinder, and a thin-walled pipe, all of mass m, roll smoothly along identical loop-the-loop tracks when released from rest along the straight section (see figure below). The circular loop has radius R, and the sphere, cylinder, and pipe have radius $r \ll R$ (i.e., the size of the objects may be neglected when compared to the other distances involved). If $h = 2.8R$, which of the objects will make it to the top of the loop? Justify your answer with an explicit calculation. The moments of inertia for the objects are listed below.

$$I = \begin{cases} \frac{2}{5}mr^2 & \text{sphere} \\ \frac{1}{2}mr^2 & \text{cylinder} \\ mr^2 & \text{pipe} \end{cases} \tag{32}$$

Hint: consider a single object with $I = kmr^2$ *to solve the general problem, and evaluate these three special cases only at the end.*



Solution: To start with, we just need to do conservation of energy. The object goes through a height $h - 2R$ to get to the top of the loop. Including both rotational and translational kinetic energy,

$$mg(h - 2R) = \frac{1}{2}mv^2 + \frac{1}{2}(kmr^2)\omega^2 = (1 + k)\left(\frac{1}{2}mv^2\right) \tag{33}$$

This doesn't tell us if the object actually makes it to the top of the loop or not. For that, we need to be sure that the velocity is high enough to be consistent with the required centripetal force. The centripetal force must be provided by the object's weight.

$$\frac{mv^2}{R} \geqslant mg \tag{34}$$

$$v^2 \geqslant Rg \tag{35}$$

Using the energy equation, we have another equation for v^2. Combining:

$$v^2 = \frac{2g\,(h - 2R)}{1 + k} \geqslant Rg \tag{36}$$

$$k \leqslant \frac{h - 2R}{R} = \frac{h}{R} - 2 \tag{37}$$

Given $h = 2.8R$, our condition is that $k \leqslant 0.8$. This is true for the sphere ($k = 2/5$) and the cylinder

($k = 1/2$), but not for the pipe ($k = 1$). Thus, the sphere and cylinder make it, but the pipe does not.

10. The rotational inertia (moment of inertia) of a collapsing spinning star drops to $\frac{1}{3}$ its initial value. What is the ratio of the new rotational kinetic energy to the initial rotational kinetic energy?

Solution: If we need the rotational kinetic energy ratio, we'll have to get the relationship between the angular velocities first. For that all we need is conservation of angular momentum, noting that the final moment of inertia I_f is one third of the initial value I_i.

$$L_i = L_f \tag{38}$$

$$I_i\omega_i = I_f\omega_f = \frac{1}{3}I_i\omega_f \tag{39}$$

$$\omega_f = 3\omega_i \tag{40}$$

Makes sense: if the moment of inertia goes down three times, the rate of rotation has to go up three times to conserve angular momentum. That's all we need to get the kinetic energy ratio.

$$\frac{K_i}{K_f} = \frac{\frac{1}{2}I_i\omega_i^2}{\frac{1}{2}I_f\omega_f^2} = \frac{1}{3} \tag{41}$$

Problems for 12 June (due 15 June)

11. The fastest possible rate of rotation of a planet is that for which the gravitational force on material at the equator just barely provides the centripetal force needed for the rotation. Show that the corresponding shortest period of rotation is

$$T = \sqrt{\frac{3\pi}{G\rho}}$$

where ρ is the uniform density (mass per unit volume) of the spherical planet. The volume of a sphere is $\frac{4}{3}\pi r^3$, where r is the radius of the sphere.

Solution: There are a few ways to go about this. Perhaps the shortest is just to use Kepler's law, which we derived from the gravitational and centripetal forces in the first place, along with the fact that the mass is $M = \frac{4}{3}\pi r^3\rho$

$$T^2 = \frac{4\pi^2}{GM} r^3 = \frac{4\pi^2}{G\frac{4}{3}\pi r^3 \rho} r^3 = \frac{3\pi}{G\rho} \tag{42}$$

$$T = \sqrt{\frac{3\pi}{G\rho}} \tag{43}$$

If you didn't think to use Kepler's law, you'd first start with $T = 2\pi r/v$ and add the centripetal force balance $mv^2/r = GMm/r^2$ (which you'd solve for v and plug in the equation for T). That will bring you to Kepler's law, at which point you proceed as above.

12. The period of the earth's rotation about the sun is 365.256 days. It would be more convenient to have a period of exactly 365 days. How should the mean distance from the sun be changed to correct this anomaly?

Solution: What you don't want to do is complicate this one with numbers right away, or it will become messy. Symbolic solution first. Start with Kepler's law, which relates period and orbital distance. Consider the present case period T_1 and orbital distance r_1, and the hypothetical 365 day year is T_2 with orbital distance r_2. Kepler's law states

$$T^2 = \frac{4\pi^2}{GM} r^3 \tag{44}$$

That means $T^2 \propto r^3$. Taking the ratio of T_1 to T_2 is perhaps the easiest thing to do.

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3} \tag{45}$$

$$r_2^3 = \left(\frac{T_2}{T_1}\right)^2 r_1^3 \tag{46}$$

$$r_2 = r_1 \sqrt[3]{\frac{T_2^2}{T_1^2}} \approx 0.9995 r_1 \tag{47}$$

Moving the earth closer to the sun by about 0.05% will do the job. But wait, you say, we don't know what r_1 is, so we don't know how much the distance has to change by! Perhaps not, but you can find r_1 from the known period $T_1 = 365.256$ days and the mass of the sun (given on the formula sheet). Solving Kepler's law for r_1,

$$r_1 = \sqrt[3]{\frac{GM_s T_1^2}{4\pi^2}} \approx 1.496 \times 10^{11}\,\text{m} \tag{48}$$

Armed with this, you should find that the sun needs to move closer to the sun by about 7×10^7 m

13. The space shuttle releases a 470 kg satellite while in an orbit 280 km above the surface of the earth. A rocket engine on the satellite boosts it to a geosynchronous orbit. How much energy is required for the orbit boost? (Note: the earth's radius is 6378 km, its mass is 5.98×10^{24} kg, and $G = 6.67 \times 10^{-11} N \cdot m^2 kg^{-2}$. Hint: "geosynchronous" means the satellite's period T is 24 hrs.)

Solution: For a geosynchronous orbit, the period T is 24 hr. Using Kepler's law, we can find the distance from the earth's center for this orbit:

$$T^2 = \frac{4\pi^2 rg^3}{GM_e} \tag{49}$$

$$r_g = \sqrt[3]{\frac{GMT^2}{4\pi^2}} \approx 42,300 \text{ km} \tag{50}$$

Thus, the satellite changes its orbit to 42,300 km starting from $h = 280$ km above the earth's surface, a distance $R_e + h$ from the earth's center. Remember that it is the distance from the earth's center, not its surface, which is important for gravitation. This will clearly change the satellite's potential energy, but its kinetic energy will also change. Fortunately, we know the total energy (kinetic plus potential) of an orbiting body of mass m is $E_{tot} = -\frac{1}{2}\frac{GM_e m}{r}$. The change in energy is thus

$$\Delta E_{tot} = -\frac{1}{2}\frac{GM_e m}{r_g} - \left(-\frac{1}{2}\frac{GM_e m}{R_e + h}\right) \approx 1.2 \times 10^{10} \text{ J} \tag{51}$$

14. Calculate the mass of the Sun given that the Earth's distance from the Sun is 1.496×10^{11}m. (Hint: you already know the period of the Earth's orbit.)

Solution: We know the earth's period of rotation T is about 365 days, or about 3.15×10^7 s. Given the earth's orbital distance r, we can use Kepler's law to find the mass of the sun M_s.

$$T^2 = \frac{4\pi^2 r^3}{GM_s} \tag{52}$$

$$M_s = \frac{4\pi^2 r^3}{GT^2} \approx 2 \times 10^{30} \text{ kg} \tag{53}$$