UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 105 LeClair Summer 2012

Problem Set 11

Instructions:

1. Answer all questions below. All questions have equal weight.
2. Due Fri 22 June 2012 at the start of lecture, or electronically by midnight.
3. You may collaborate, but everyone must turn in their own work.

1. An aircraft door closes by pushing it inside the airplane first. We will assume $P = 0$ outside the aircraft, and $P = 0.9\,\text{atm}$ inside during flight. If the sealing surface of the door is 5 cm wide all around the door, and the door's outer dimensions are 2 m by 0.7 m, what is the total force required to open the door while in flight?

Solution: The force will be the sealing surface area times the pressure difference applied to it. The sealing surface one can consider to be two strips of 2 m by 0.05 m and two strips of 0.6 m by 0.05 m, giving a net area of $0.26\,\text{m}^2$.[i] The pressure difference is $P = 0.9\,\text{atm} \approx 9.12 \times 10^4\,\text{Pa}$. Thus,

$$F = \Delta P A \approx \left(9.12 \times 10^4\,\text{Pa}\right)\left(0.26\,\text{m}^2\right) \approx 2.4 \times 10^4\,\text{N} \tag{1}$$

For comparison, note that the gravitational force required to lift a 1000 lb weight is only about 4.4×10^3 N. In other words, even with a 5 cm sealing surface, there is no way you're going to open that door at altitude while the cabin is pressurized.

2. Viscosity of most fluids can be represented by an extra "drag" force on a body moving in a liquid. For a body of spherical shape, the drag force is reasonably well approximated by $F_{drag} = 6\pi\eta R v$, where v is the velocity of the body and η is a parameter of the fluid. The presence of viscosity leads to a "terminal velocity" of a body falling in a fluid (*e.g.,* a person falling in air).

Consider a sphere of radius R and density ρ_s falling through a fluid of density ρ and viscosity parameter η. Find an expression for the terminal velocity of the sphere.

Solution: Terminal velocity is when the the object in question reaches a constant maximum velocity, which must be when the net force on the object is zero. Physically, the object's speed has become so high, and the corresponding drag force so great that it manages to balance the object's weight and any other forces. The weight of a sphere of radius R and density ρ_s is

$$F_w = \frac{4}{3}\pi R^3 \rho_s g \tag{2}$$

[i] If you came up with a slightly different area, that is not a problem - the geometry is not strictly defined in the problem, so you had a choice to either add or subtract 5 cm from the given dimensions. Either way is fine, it does not change the conclusion.

If we are in a surrounding fluid of density ρ, we must also account for the buoyant force, equal to the weight of the displaced fluid. This is the same as the expression above if we substitute $\rho_s \rightarrow \rho$

$$B = \frac{4}{3}\pi R^3 \rho g \tag{3}$$

The drag and buoyant forces will act in one direction, the weight of the object opposing them. A force balance yields, at terminal velocity,

$$0 = F_d + B - F_w \tag{4}$$
$$0 = 6\pi\eta R v + \frac{4}{3}\pi R^3 \rho g - \frac{4}{3}\pi R^3 \rho_s g \tag{5}$$
$$6\pi\eta R v = \frac{4}{3}\pi R^3 \left(\rho_s - \rho\right) g \tag{6}$$
$$v = \frac{2g}{9\eta} R^2 \left(\rho_s - \rho\right) \tag{7}$$

The fact that terminal velocity depends on particle size has many interesting technological applications. (As a quick for-instance: `http://en.wikipedia.org/wiki/Fluidized_bed_reactor`.)

3. A pendulum is formed by pivoting a long thin rod of mass M and length L about a point on the rod. If the pivot is a distance x from the rod's center, for what x is the period of the pendulum minimum? The moment of inertia for a thin rod about its center of mass is $I = \frac{1}{12}ML^2$.

Solution: In the end, we only have a physical pendulum, and we already know that the period is given by

$$T = 2\pi\sqrt{\frac{I}{mgh}}$$

where I is the moment of inertia of the rod (of mass m) about the pivot point, and h is the distance between the rod's center of mass and the pivot point. Let the pivot be a distance x from the end of the rod, making it a distance $l/2-x$ from the center of mass. The moment of inertia is then

$$I = I_{com} + m\left(\frac{l}{2} - x\right)^2 = \frac{1}{12}ml^2 + m\left(\frac{l}{2} - x\right)^2$$

The distance between the center of mass and the pivot is $h = l/2 - x$, so

$$I = \frac{1}{12}ml^2 + mh^2$$

The period is thus

$$T = 2\pi\sqrt{\frac{\frac{1}{12}l^2 + h^2}{gh}} = 2\pi\sqrt{\frac{l^2}{12gh} + \frac{h}{g}}$$

We wish to find x such that T is a maximum, which means $dT/dx=0$. Noting that $dT/dx=-dT/dh$,

$$\frac{dT}{dx} = -\frac{dT}{dh} = 0 \tag{8}$$

$$\frac{d}{dh}\left[2\pi\sqrt{\frac{\frac{1}{12}l^2 + h^2}{gh}}\right] = 0 \tag{9}$$

$$2\pi\left(\frac{1}{2}\right)\left(\frac{-l^2}{12gh^2} + \frac{1}{g}\right)\left(\frac{\frac{1}{12}l^2 + h^2}{gh}\right)^{-1/2} = 0 \tag{10}$$

$$\Longrightarrow \quad \frac{-l^2}{12gh^2} + \frac{1}{g} = 0 \tag{11}$$

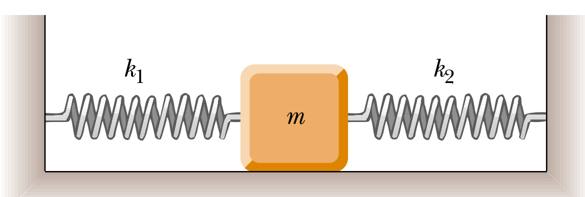
$$12h^2 = l^2 \tag{12}$$

$$h = \frac{l}{2\sqrt{3}} \approx 0.29l \tag{13}$$

A quick second derivative test or a plot of dT/dh verifies that this is indeed a minimum, not a maximum. The minimum period is therefore

$$T_{min} = T\Big|_{h=\frac{l}{2\sqrt{3}}} = 2\pi\sqrt{\frac{\frac{1}{12}l^2 + \frac{1}{12}l^2}{g\frac{l}{2\sqrt{3}}}} = 2\pi\sqrt{\frac{l}{\sqrt{3}g}} \approx 2.26\,\text{s}$$

4. A block of mass m is connected to two springs of force constants k_1 and k_2 as shown below. The block moves on a frictionless table after it is displaced from equilibrium and released. Determine the period of simple harmonic motion. (Hint: what is the total force on the block if it is displaced by an amount x?



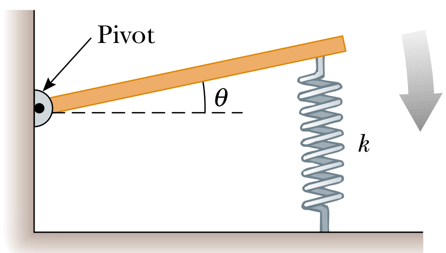
Solution: Say we displace the block to the right by an amount x. Both springs will try to bring the block back toward equilibrium - one will pull, one will push, but *both will act in the same direction.* That means the net force is

$$F_{net} = -k_1x - k_2x = -(k_1 + k_2)x = ma \tag{14}$$

This is exactly the same as what we would find for a single spring, except the spring constant has become $k_1 + k_2$ rather than just k. The solution must be

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k_1 + k_2}} \tag{15}$$

5. A horizontal plank of mass m and length L is pivoted at one end. The plank's other end is supported by a spring of force constant k. The moment of inertia of the plank about the pivot is $I = \frac{1}{3}mL^2$. The plank is displaced by a small angle θ from horizontal equilibrium and released. Find the angular frequency ω of simple harmonic motion. (Hint: consider the torques about the pivot point.)



Solution: The presence of a pivot point – even labeled as such – immediately suggests the use of torque to solve this problem. First, we need to find the compression of the spring at equilibrium, i.e., $\theta = 0$. Since the plank has non-zero mass, even without an angular displacement the spring must be compressed by some amount at equilibrium. Once we have found the equilibrium position, we can worry about the torques when a small angular displacement θ is applied.

Let counterclockwise rotations be defined as positive, and let the equilibrium position of the spring correspond to the tip of the plank being at vertical position x_o relative to its unstretched length. The sum of the torques about the pivot point at equilibrium ($\theta = 0$) is given by considering the weight of the plank acting about its center of mass and the restoring force of the spring. The plank may be treated as a point mass a distance $L/2$ from the pivot point, while the spring force acts at a distance L. At equilibrium, the net torque must be zero.

$$\sum \tau = -mg\left(\frac{L}{2}\right) + kx_oL = 0 \tag{16}$$

$$x_o = \frac{mg}{2k} \tag{17}$$

Now that we have the equilibrium position, we can find the torques when the plank makes an angle θ with the vertical. Since the plank cannot change its length (we assume), the amount that the spring stretches should correspond to the arc length that the tip of the plank moves through, $L\theta$, if the angle is relatively small.[ii] The spring is therefore displaced by an amount $L\theta - x_o$ from its unstretched length. This gives us the spring's restoring force. Since the spring is attached to the plank, the spring force always acts perpendicularly to the length of the plank at distance L, and the torque is easily found.

The plank's weight still acts at a distance $L/2$ from the center of mass, but now at an angle $90 - \theta$ relative to the axis of the plank. The overall torque is then

$$\sum \tau = -mg\left(\frac{L}{2}\right)\sin(90-\theta) - k\left(L\theta - x_o\right)L = -\frac{1}{2}mgL\cos\theta - kL^2\theta + kx_oL$$

Since the angle θ is small, we may approximate $\cos\theta \approx 1$, and we may also make use of our earlier expression for x_o. Finally, out of equilibrium the torques must give the moment of inertia times the angular acceleration.

$$\sum \tau = -\frac{1}{2}mgL\cos\theta - kL^2\theta + kx_oL \approx -\frac{1}{2}mgL - kL^2\theta + \frac{1}{2}mgL = -kL^2\theta = I\alpha \tag{18}$$

$$-kL^2\theta = I\frac{d^2\theta}{dt^2} \tag{19}$$

Noting that $I = \frac{1}{3}mL^2$ for a thin plank, we can put the last equation in the desired form for simple harmonic motion in terms of known quantities:

$$\frac{d^2\theta}{dt^2} = -\frac{3k}{m}\theta \tag{20}$$

$$\omega = \sqrt{\frac{3k}{m}} \tag{21}$$

Note that the length of the plank does not matter at all.

[ii] Small enough such that we don't have to worry about the spring bending to the left, for one.