University of Alabama
Department of Physics and Astronomy

PH 105 LeClair — Summer 2012

Problem Set 3 Solutions

Instructions:

1. Answer all questions below. All questions have equal weight.
2. Show your work for full credit; using the problem template is recommended.
3. All problems are due Friday 1 June 2012 at the start of lecture.
4. You may collaborate, but everyone must turn in their own work.

1. *HRW 4.15* A particle leaves the origin with an initial velocity $\vec{\mathbf{v}}_i = (3.00\,\hat{\boldsymbol{\imath}})\,\mathrm{m/s}$ and a constant acceleration $\vec{\mathbf{a}} = (-1.00\,\hat{\boldsymbol{\imath}} - 0.500\,\hat{\boldsymbol{\jmath}})\,\mathrm{m/s^2}$. When it reaches its maximum x coordinate, what are its (a) velocity and (b) position vectors?

Solution: **Given:** The initial velocity and acceleration vectors of a particle. Given in vector form, we may write them explicitly in terms of components as well:

$$\vec{\mathbf{v}}_i = 3.00\,\hat{\boldsymbol{\imath}}\mathrm{m/s} = v_{ix}\,\hat{\boldsymbol{\imath}} + v_{iy}\,\hat{\boldsymbol{\jmath}} \tag{1}$$

$$\Longrightarrow \qquad v_{ix} = 3.00\,\mathrm{m/s} \qquad v_{iy} = 0 \tag{2}$$

$$\vec{\mathbf{a}} = (-1.00\,\hat{\boldsymbol{\imath}} - 0.500\,\hat{\boldsymbol{\jmath}})\,\mathrm{m/s^2} = a_{ix}\,\hat{\boldsymbol{\imath}} + a_{iy}\,\hat{\boldsymbol{\jmath}} \tag{3}$$

$$\Longrightarrow \qquad a_{ix} = -1.00\,\mathrm{m/s^2} \qquad a_{iy} = -0.500\,\mathrm{m/s^2} \tag{4}$$

Find: The maximum x coordinate, and the velocity and position vectors at the corresponding time.

Sketch: A sketch may be a bit unnecessary in this case, but here you are:

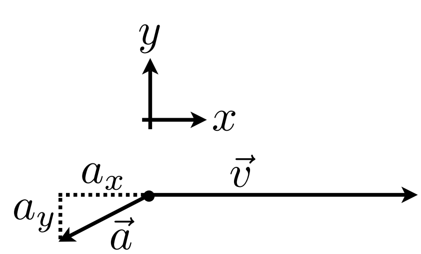


Figure 1: *A particle at the origin with an initial velocity and acceleration.*

Relevant equations: Since the acceleration is constant, our usual equations of motion work for the motion along both x and y axes. We know that we have a constant x component of acceleration $a_x = -1.00\,\mathrm{m/s^2}$, an initial velocity along the x axis of $v_{xi} = 3.00\,\mathrm{m/s^2}$, and an initial position of

$x_i = 0$ since the particle starts at the origin. The velocity and x coordinate are easily found at any time t later:

$$v_x(t) = v_{ix} + a_x t \tag{5}$$
$$x(t) = x_i + v_{ix} t + \frac{1}{2} a_x t^2 \tag{6}$$

Since our particle starts out at the origin, $x_i = 0$. Similiarly, we can write down the y velocity and position at any later time:

$$v_y(t) = v_{iy} + a_y t \tag{7}$$
$$y(t) = y_i + v_{iy} t + \frac{1}{2} a_y t^2 \tag{8}$$

Again, since we start at the origin, $y_i = 0$.

Now that we have $x(t)$, we can find the time at which it is maximum, t_{max}: by differentiation:

$$\left.\frac{dx}{dt}\right|_{t_{max}} = 0 \qquad \text{and} \qquad \left.\frac{d^2x}{dt^2}\right|_{t_{max}} < 0 \tag{9}$$

Next, we need to find the position vector $\vec{\mathbf{r}}(t)$ and velocity vector $\vec{\mathbf{v}}(t)$:

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t + \frac{1}{2}\vec{\mathbf{a}} t^2 = (x_i\,\hat{\imath} + y_i\,\hat{\jmath}) + (v_{ix}\,\hat{\imath} + v_{iy}\,\hat{\jmath})t + \frac{1}{2}(a_x\hat{\imath} + a_y\hat{\jmath})t^2 \tag{10}$$
$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t = (v_{ix}\,\hat{\imath} + v_{iy}\,\hat{\jmath}) + (a_x\hat{\imath} + a_y\hat{\jmath})t \tag{11}$$

Again, since the particle starts at the origin, $\vec{\mathbf{r}}_i = 0$; we know all the other quantities already. Finally, we need to evaluate $\vec{\mathbf{v}}$ and $\vec{\mathbf{r}}$ at t_{max} to complete the problem.

Symbolic solution: First, we can immediately write down the equation for $x(t)$ and maximize it:

$$\begin{aligned}
&x(t) = x_i + v_{ix} t + \frac{1}{2} a_x t^2 \\
&\left.\frac{dx}{dt}\right|_{t_{max}} = v_{ix} + a_x t_{max} = 0 \quad \Longrightarrow \quad t_{max} = -\frac{v_{ix}}{a_x} \\
&\frac{d^2x}{dt^2} = a_x
\end{aligned} \tag{12}$$

The time found this way will correspond to a maximum (rather than a minimum) if the second derivative $d^2x/dt^2 = a_x$ is negative, i.e., so long as the acceleration a_x is negative. Since we know

$a_x = -1.00\,\mathrm{m/s}^2$, we have definitely found a maximum. Given the time at which the maximum x coordinate occurs, we can find that x coordinate readily:

$$\begin{aligned}
\vec{v}(t_{max}) &= \vec{v}_i + \vec{a}t_{max} = \vec{v}_i - \frac{v_{ix}}{a_x}\vec{a} \\
\vec{r}(t_{max}) &= \vec{r}_i + \vec{v}_i t_{max} + \frac{1}{2}\vec{a}t_{max}^2 = \vec{r}_i - \frac{v_{ix}}{a_x}\vec{v}_i + \frac{v_{ix}^2}{2a_x^2}\vec{a}
\end{aligned} \tag{13}$$

Numeric solution: The x components of the velocity and acceleration are

$$\begin{aligned}
a_x &= -1.00\,\mathrm{m/s}^2 \\
v_x &= 3.00\,\mathrm{m/s}
\end{aligned} \tag{14}$$

which makes the time for the maximum x coordinate

$$t_{max} = -\frac{v_{ix}}{a_x} = 3.00\,\mathrm{s} \tag{15}$$

Note also that $d^2x/dt^2 = -1.00$, which ensures that we have found a maximum of $x(t)$. At this point, we can just write down the expressions for $\vec{r}(t)$ and $\vec{v}(t)$ and plug in the numbers:

$$\begin{aligned}
\vec{v}(t_{max}) &= \vec{v}_i + \vec{a}\,t_{max} \\
&= (3.00\,\mathrm{m/s}\,\hat{\imath}) + \left(-1.00\,\mathrm{m/s}^2\,\hat{\imath} - 0.500\,\mathrm{m/s}^2\,\hat{\jmath}\right)(3.00\,\mathrm{s}) = (-1.5\,\hat{\jmath})\ \mathrm{m/s}
\end{aligned} \tag{16}$$

$$\begin{aligned}
\vec{r}(t_{max}) &= \vec{r}_i + \vec{v}_i t_{max} + \frac{1}{2}\vec{a}\,t_{max}^2 \\
&= 0 + (3.00\,\mathrm{m/s}\,\hat{\imath})(3.00\,\mathrm{s}) + \frac{1}{2}\left(-1.00\,\mathrm{m/s}^2\,\hat{\imath} - 0.500\,\mathrm{m/s}^2\,\hat{\jmath}\right)(3.00\,\mathrm{s})^2 \\
&= (9.00\,\mathrm{m}\,\hat{\imath}) + (-4.50\,\mathrm{m}\,\hat{\imath} - 2.25\,\mathrm{m}\,\hat{\jmath}) = (4.5\,\hat{\imath} - 2.25\,\hat{\jmath})\ \mathrm{m}
\end{aligned} \tag{17}$$

Double check: We carried the units throughout our calculations, and can be fairly confident that they are correct. We also performed the second derivative test to ensure that we found a maximum in $x(t)$.

2. *HRW 4.23* A projectile is fired horizontally from a gun that is 45.0 m above flat ground, emerging from the gun with a speed of 250 m/s. (a) How long does the projectile remain in the air? (b) At what horizontal distance from the firing point does it strike the ground? (c) What is the magnitude of the vertical component of its velocity as it strikes the ground?

Solution: Since the particle is launched purely horizontally, there is no initial velocity in the vertical direction. The vertical motion is completely dictated by the gravitational acceleration. For the sake of concreteness, define the $+y$ axis to be "up" (making gravitational acceleration in the

$-y$ direction) and the $+x$ axis to be in the direction of the projectile's motion. Let the origin be at the position on the ground directly below he gun (so $x = 0$, $y = 45.0\,\mathrm{m}$ are the launch coordinates). Call the initial height $h = 45.0\,\mathrm{m}$ and the initial velocity $v_i = 250\,\mathrm{m/s}$.

Since the vertical motion is independent of the horizontal motion, and completely determined by gravitational acceleration, the projectile will remain in the air exactly as long as it takes the projectile to fall from height h, irrespective of the horizontal launch speed. The equation for $y(t)$ can be written down immediately, and the time at which $y = 0$ is when the projectile hits the ground. We know the initial height is $y_i = h$ and the initial vertical velocity is $v_{iy} = 0$, so:

$$y(t) = y_i + v_{iy}t - \frac{1}{2}gt^2 = h - \frac{1}{2}gt^2 = 0 \tag{18}$$

$$\implies t_{\text{hit}} = \sqrt{\frac{2h}{g}} \approx 3.03\,\mathrm{s} \tag{19}$$

How far does the projectile travel horizontally? We know its horizontal velocity, and we know it has only 3.03 s to travel before it hits the ground. Given $x_i = 0$ and $v_{ix} = 250\,\mathrm{m/s}$, and knowing we have no acceleration along the x axis, we can write down the x coordinate as a function of time and then evaluate at the time t_{hit} we just found:

$$x(t) = x_i + v_{ix}t + \frac{1}{2}a_x t^2 = v_{ix}t \tag{20}$$

$$\implies x_{\text{hit}} = v_{ix}\sqrt{\frac{2h}{g}} \approx 757\,\mathrm{m} \tag{21}$$

Finally, the vertical velocity just before the projectile hits the ground is the same as it would be for any other object dropped from height h. We need only evaluate v_y at the time t_{hit}.

$$v_y(t) = v_{iy} - gt = -gt \tag{22}$$

$$\implies v_{y,\text{hit}} = -g\sqrt{\frac{2h}{g}} = -\sqrt{2gh} \approx 29.7\,\mathrm{m/s} \tag{23}$$

Negative in this case because the projectile is moving in the $-y$ direction.

3. *HRW 4.35* A rifle that shoots a bullet at $460\,\mathrm{m/s}$ is to be aimed at a target $45.7\,\mathrm{m}$ away. If the center of the target is level with the rifle, how high above the target must the rifle barrel be pointed so that the bullet hits dead center?

Solution: **Given:** The magnitude of the initial velocity of a fired bullet v_i and its distance from a target d.

Find: The height above the target that the shooter must aim y_{aim}. We can easily find this once we know the firing angle θ required for the bullet to hit the target. That is, the angle such that the bullet is at the same height a distance d from where it is fired.

Sketch: For convenience, let the origin be at the position the bullet is fired from. Let the $+x$ axis run horizontally, from the bullet to the target, and let the $+y$ axis run vertically. Let time $t\!=\!0$ be the moment the projectile is launched.

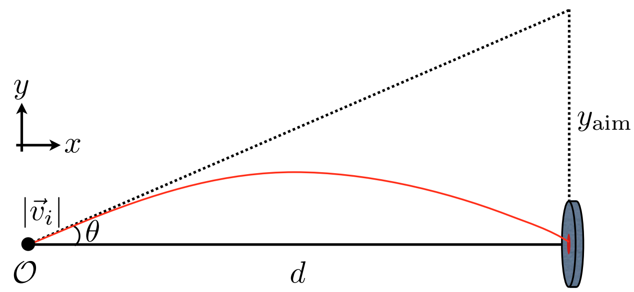


Figure 2: *Firing a rifle at a distant target. The bullet's trajectory is (approximately) shown in red.*

The bullet is fired at an initial velocity $|\vec{\mathbf{v}}_i|$ and angle θ, a distance d from a target. The target is at the same vertical position as the rifle, so we need to find the angle θ and resulting y_{aim} such that the bullet is at $y\!=\!0$ at $x\!=\!d$.

Relevant equations: In the x direction, we have constant velocity and no acceleration, with position starting at the origin at $t\!=\!0$:

$$x(t) = v_{ix}t = |\vec{\mathbf{v}}_i|t\cos\theta \tag{24}$$

In the y direction, we have an initial constant velocity of $v_{iy}\!=\!|\vec{\mathbf{v}}_i|\sin\theta$ and a constant acceleration of $a_y\!=\!-g$:

$$y(t) = v_{iy}t - \frac{1}{2}gt^2 \tag{25}$$

Solving Eq. 24 for t and substituting into Eq. 25 yields our general projectile equation, giving the path of the projectile $y(x)$ when launched from the origin with initial velocity $|\vec{\mathbf{v}}_i|$ and angle θ above the x axis:

$$y(x) = x\tan\theta - \frac{gx^2}{2|\vec{\mathbf{v}}_i|^2\cos^2\theta} \tag{26}$$

With our chosen coordinate system and origin, $y_o = 0$. We also need the aiming height above the target in terms of the target distance and firing angle, which we can get from basic trigonometry:

$$\tan\theta = \frac{y_{\text{aim}}}{d} \tag{27}$$

Note that one can also use the "range equation" directly, but this is less instructive. It is fine for you to do this in your own solutions, but keep in mind you will probably not be given these sort of specialized equations on an exam – you should know how to derive them.

Symbolic solution: We desire the bullet to reach point $(d, 0)$. Substituting these coordinates into Eq. 26, and solving for θ:

$$y(x) = x\tan\theta - \frac{gx^2}{2|v_i|^2\cos^2\theta} \tag{28}$$

$$0 = d\tan\theta - \frac{gd^2}{2|v_i|^2\cos^2\theta} \tag{29}$$

$$d\tan\theta = \frac{gd^2}{2|v_i|^2\cos^2\theta} \tag{30}$$

$$\tan\theta\cos^2\theta = \frac{gd}{2|\vec{v}_i|^2} \tag{31}$$

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta = \frac{gd}{2|\vec{v}_i|^2} \tag{32}$$

$$\implies \theta = \frac{1}{2}\sin^{-1}\left[\frac{gd}{|\vec{v}_i|^2}\right] \tag{33}$$

Given θ, rearranging Eq. 27 gives us the aiming height:

$$y_{\text{aim}} = d\tan\theta \tag{34}$$

Numeric solution: We are given $|\vec{v}_i| = 460\,\text{m/s}$ and $d = 45.7\,\text{m}$:

$$\theta = \frac{1}{2}\sin^{-1}\left[\frac{gd}{|\vec{v}_i|^2}\right] = \frac{1}{2}\sin^{-1}\left[\frac{\left(9.81\,\text{m/s}^2\right)\left(4.57\,\text{m}\right)}{\left(460\text{m/s}\right)^2}\right] \approx 0.06069^\circ \tag{35}$$

Given the angle, we can find the height above the target we need to aim:

$$y_{\text{aim}} = d\tan\theta \approx (45.7\,\text{m})\tan\left(0.06069^\circ\right) \xrightarrow[\text{digits}]{\text{sign.}} 0.0484\,\text{m} = 4.84\,\text{cm} \tag{36}$$

Double check: One check is use the pre-packaged projectile range equation and make sure that we get the same answer. Given $\theta \approx 0.0607°$, we should calculate a range of d.

$$R = \frac{|\vec{v}_i|^2 \sin 2\theta}{g} = \frac{(460\,\mathrm{m/s})\,(\sin 0.1214°)}{(9.81\,\mathrm{m/s^2})} \approx 45.7\,\mathrm{m} \tag{37}$$

This is not truly an independent check, since it is derived using the same equations we used above, but it is a nice indication that we haven't gone wrong anywhere.

As a more independent estimate, we can first calculate the time it would take the bullet to reach the target in the absence of gravitational acceleration – if it were just heading straight toward the target at 460 m/s. This is not so far off the real time, since the firing angle is small anyway:

$$t_{\mathrm{est}} = \frac{d}{|\vec{v}_i|} \approx 0.1\,\mathrm{s} \tag{38}$$

In that time, how far would the bullet fall under the influence of gravity (alone)?

$$y_{\mathrm{fall}} \approx -\frac{1}{2} g t_{\mathrm{est}}^2 \approx 0.05\,\mathrm{m} \tag{39}$$

Thus, we estimate that the bullet should fall about 5 cm on its way to the target, meaning we should aim about 5 cm high, in line with what we calculate by more exact means.

You can also verify that units come out correctly in Eq. 35 and Eq. 36. The argument of the $\sin^{-1}$ function must be dimensionless, as it is, and y_{aim} should come out in meters, as it does. If you carry the units through the entire calculation, or at least solve the problem symbolically, without numbers until the last step, this sort of check is trivial.

4. A batter hits a baseball coming off of the bat at a 45° angle, making contact a distance 1.22 m above the ground. Over level ground, the batted ball has a range of 107 m. Will the ball clear a 7.32 m tall fence at a distance of 97.5 m? Justify your answer. *Hint: use the range equation to get the velocity, then use the trajectory equation to find the path of the ball.*

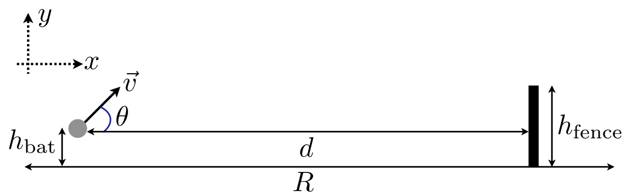
Solution: This is problem 4.47 from your textbook.

Find: Whether a batted baseball clears a fence, and by what amount it does or does not.

Given: The baseball's initial launch height and angle, the range the baseball would have without the fence, the distance to the fence and its height.

Sketch: Let the y axis run vertically and the x axis horizontally as shown below. Let the range the baseball would have without the fence be $R = 107\,\mathrm{m}$, with the distance to the fence $d = 97.5\,\mathrm{m}$

and its height $h_{\text{fence}} = 7.32\,\text{m}$. The baseball is batted at an angle $\theta = 45°$ at speed v_i a height of $h_{\text{bat}} = 1.22\,\text{m}$ above the ground.



Let the origin be at the position the ball leaves the bat. The height of the fence *relative to the height of the bat* is then

$$\delta h = h_{\text{fence}} - h_{\text{bat}}$$

What we really need to determine is the ball's y coordinate at $x = d$. If $y > \delta h$, the ball clears the fence. We can use the range the baseball would have without the fence and the launch angle to find the ball's speed, which will allow a complete calculation of the trajectory.

Relevant equations: We need only the equations for the range and trajectory of a projectile over level ground:

$$R = \frac{v_i^2 \sin 2\theta}{g}$$

$$y(x) = x \tan\theta - \frac{gx^2}{2v_i^2 \cos^2\theta}$$

Symbolic solution: From the range equation above, we can write the velocity in terms of known quantities:

$$v_i = \sqrt{\frac{Rg}{\sin 2\theta}}$$

The trajectory then becomes

$$y(x) = x \tan\theta - \frac{gx^2 \sin 2\theta}{2Rg \cos^2\theta} = x \tan\theta - \frac{x^2 \sin 2\theta}{2R \cos^2\theta}$$

The height difference between the ball and the fence is $y(d) - \delta h$. If it is positive, the ball clears the fence.

$$\text{clearance} = y(d) - \delta h = d \tan\theta - \frac{d^2 \sin 2\theta}{2R \cos^2\theta} - \delta h = d \tan\theta - \frac{d^2 \sin 2\theta}{2R \cos^2\theta} - h_{\text{fence}} + h_{\text{bat}}$$

Numeric solution: Using the numbers given, and noting $\tan 45° = 1$, $\sin 90° = 1$, and $\cos^2 45° = 1/2$

$$\text{clearance} = \mathsf{d} - \frac{\mathsf{d}^2}{\mathsf{R}} - \mathsf{h}_{\text{fence}} + \mathsf{h}_{\text{bat}} = 97.5\,\mathrm{m} - \frac{(97.5\,\mathrm{m})^2}{107\,\mathrm{m}} - 7.32\,\mathrm{m} + 1.22\,\mathrm{m} \approx 2.56\,\mathrm{m}$$

The ball does clear the fence, by approximately 2.56 m.