University of Alabama
Department of Physics and Astronomy

PH 105 LeClair Summer 2012

Problem Set 7 Solutions

1. The potential energy of an Argon dimer may be modeled by

$$U(r) = 4\epsilon\left(\frac{\sigma^{12}}{r^{12}} - \frac{\sigma^6}{r^6}\right)$$

(a) Find the equilibrium separation of the dimer (i.e., the value of r at equilibrium).
(b) Is the equilibrium stable? Justify your answer.

Solution: Equilibrium is defined by a net zero force, or equivalently, a minimum in potential energy. Either way,

$$-F = \frac{dU}{dr} = 4\epsilon\left(-\frac{12\sigma^{12}}{r^{13}} + \frac{6\sigma^6}{r^7}\right) = 0 \tag{1}$$

Solving,

$$\frac{12\sigma^{12}}{r^{13}} = \frac{6\sigma^6}{r^7} \tag{2}$$

$$\frac{2\sigma^6}{r^6} = 1 \tag{3}$$

$$r_{eq} = \sqrt[6]{2}\sigma \tag{4}$$

To see whether it is stable or not, we should see if the $U(r)$ curve is concave upward (stable) or downward (not). This means applying the second derivative test. We know $\frac{dU}{dr}\big|_{r_{eq}}=0$, and if it is also true that $\frac{d^2U}{dr^2}\big|_{r_{eq}}<0$, we have found a minimum at r_{eq} and the equilibrium is stable.

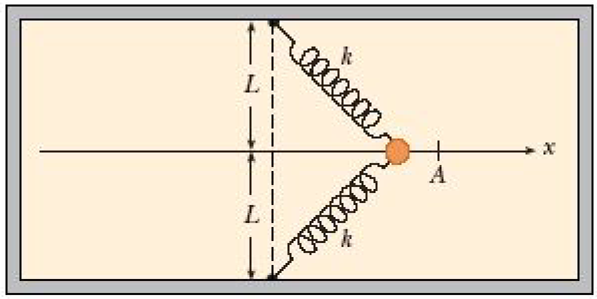
$$\frac{d^2U}{dr^2} = 4\epsilon\left(\frac{13\cdot 12\sigma^{12}}{r^{14}} - \frac{7\cdot 6\sigma^6}{r^8}\right) \tag{5}$$

$$\frac{d^2U}{dr^2}\bigg|_{r_{eq}} = 4\epsilon\left(\frac{13\cdot 12\sigma^{12}}{2^{7/3}\sigma^{14}} - \frac{7\cdot 6\sigma^6}{2^{4/3}\sigma^8}\right) = 4\epsilon\left(\frac{13\cdot 12}{2^{7/3}\sigma^2} - \frac{7\cdot 6}{2^{4/3}\sigma^2}\right) \tag{6}$$

$$= \frac{24\epsilon}{2^{4/3}\sigma^2}\left(\frac{26}{8} - 7\right) = \frac{24\epsilon}{2^{4/3}\sigma^2}\left(-\frac{15}{4}\right) < 0 \tag{7}$$

We don't even need to know σ or ϵ: the result is negative so long as they are both positive constants, so $\frac{d^2U}{dr^2}\big|_{r_{eq}}<0$ is true everywhere. Thus, we have found a stable equilibrium at r_{eq}.

2. Consider the setup below with two springs connected to a mass on a *frictionless* table. Find an expression for the potential energy as a function of the displacement along the x axis, $U(x)$. (Hint: consider the limiting cases $L \to 0$ and $x \to 0$ to check your solution. Also note that $F = -\frac{dU}{dx}$ ...)



Top view

Solution: Since potential energy is a scalar, we can just add the potential energies for the two springs together. Since the two springs are identical, we can just figure out the potential energy of one of them and double it.

We can proceed two ways: calculate the forces in the x direction, and integrate to get $U(x)$, or noting that $U_{\text{spring}} = \frac{1}{2}k\,(\Delta x)^2$ and calculating $U(x)$ directly. We will take the latter approach, since it seems dramatically less tedious.

When $x = 0$, both springs have a length L. As soon as we pull on the mass and move it to some $x \neq 0$, we can find the new length of the spring from simple geometry as $\sqrt{x^2 + L^2}$. The difference between these two lengths is how much the spring is stretched, and this difference Δx gives us $U(x)$:

$$
\begin{aligned}
U(x) &= U_{\text{spring1}} + U_{\text{spring2}} \\
&= 2U_{\text{spring1}} = 2\left(\frac{1}{2}k\,(\Delta x)^2\right) = k\,(\Delta x)^2 \\
&= k\left(\sqrt{x^2 + L^2} - L\right)^2 = kL^2 + kx^2 - 2kl\sqrt{x^2 + L^2} \\
&= kx^2 + 2kL\left(L - \sqrt{L^2 + x^2}\right)
\end{aligned}
$$

We can note that as $x \to 0$, we get $U \to 0$, which makes sense as the springs are not stretched when $x = 0$. If we let $L \to 0$, the springs are horizontal, and only stretched by a distance x, and the energy is just $2 \times \frac{1}{2}kx^2$ as it must be.

3. A block having a mass of 0.80 kg is given an initial velocity of $v_A = 1.2\,\text{m/s}$ to the right and collides with a spring of negligible mass and force constant $k = 50\,\text{N/m}$. Assuming the surface to be frictionless, what is the maximum compression of the spring after the collision?

Solution: Conservation of energy once again. The initial kinetic energy of the block is converted into potential energy stored in the spring. At the point when the spring has "absorbed" all the mass' initial kinetic energy, the mass momentarily stops before the spring begins pushing it back.

$$\begin{aligned}
K_i + U_i &= K_f + U_f \\
\frac{1}{2}mv_A^2 + 0 &= 0 + \frac{1}{2}k\,(\Delta x)^2 \\
\frac{m}{k}v_A^2 &= (\Delta x)^2 \\
\Delta x &= \sqrt{\frac{m}{k}}v_A = \sqrt{\frac{0.8\,\mathrm{kg}}{50\,\mathrm{N/m}}}\,(1.2\,\mathrm{m/s}) \approx 0.15\,\mathrm{m}
\end{aligned}$$