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PH 105 LeClair Summer 2012

Problem Set 9 Solutions

1. A cockroach with mass m rides on a disk of mass $6.00m$ and radius R. The disk rotates like a merry-go-round around its central axis at angular speed $\omega_i = 1.50\,\text{rad/s}$. The cockroach is initially at radius $r = 0.800R$, but then it crawls out to the rim of the disk. Treat the cockroach as a particle. What then is the angular speed?

Solution: The key to this problem is conservation of angular momentum. Strictly speaking, we can't use conservation of energy. In order to walk across the disc the roach must be doing work, and it would then not be true that the initial and final kinetic energies are the same - they will not be. Instead, we need to recognize that when the roach moves to the edge of the disc, the system's total mass is now distributed farther from the center of rotation. This changes the moment of inertia, which must then change the angular momentum.

First, we can write the moment of inertia in the initial and final state. The moment of inertia of a disc is well known, and we may treat the roach as a point particle. Thus, we need only know the radius of the disc, the position of the roach, and the masses of the disc and roach. For now, let the disc have mass M and the roach mass m, with the roach's initial position as βR (so $\beta = 0.800$).

$$I_{\text{total}} = I_{\text{disc}} + I_{\text{roach}} \tag{1}$$

$$I_i = \frac{1}{2}MR^2 + M\beta^2R^2 \tag{2}$$

$$I_f = \frac{1}{2}MR^2 + MR^2 \tag{3}$$

Angular momentum is $L = I\omega$. Since the moment of inertia changes from the initial to final state, for angular momentum to be conserved it is clear that ω must change as well. Let the initial and final angular velocities be ω_i and ω_f, respectively. Conservation of angular momentum implies:

$$L_i = I_i\omega_i = \left(\frac{1}{2}MR^2 + M\beta^2R^2\right)\omega_i = L_f = \left(\frac{1}{2}MR^2 + MR^2\right)\omega_f \tag{4}$$

$$\implies \qquad \omega_f = \frac{\frac{1}{2}MR^2 + m\beta^2R^2}{\frac{1}{2}MR^2 + mR^2}\omega_i \tag{5}$$

Given $\beta = 0.800$, $M = 6.00m$, and $\omega_i = 1.50\,\text{rad/s}$, we find $\omega_f \approx 1.37\,\text{rad/s}$.

A conservation of energy-based approach would get right the fact that the speed of the disc decreases owing to an increase of the moment of inertia. What it would neglect is the fraction of the initial energy that would be 'used' as work done by the roach on the disc in moving toward the rim. Neglecting this ascribes more kinetic energy to the disc and roach in the final situation than is warranted, and we would overestimate the angular velocity. Since the mass of the roach is comparatively small, the error in neglecting the work done by the roach is not severe, of order (m/M) or a bit over 15%.

2. A long uniform rod of length L and mass M is pivoted about a horizontal, frictionless pin through one end. The rod is released from rest in a vertical position. At the instant the rod is horizontal, find its angular speed. The moment of inertia of a solid rod about its center of mass is $I = \frac{1}{12}ML^2$.

Solution: In this case, conservation of energy is the way to go - gravitational potential energy is being converted to rotational kinetic energy. The change in gravitational potential energy can be found by treating the rod as a point particle of mass M located at the rod's center of mass, and seeing how the center of mass height changes. The center of mass of the rod changes by $L/2$ going from vertical to horizontal, so the change in gravitational potential energy is just $MgL/2$.

This gravitational energy will be converted to rotational kinetic energy, which we know is just $\frac{1}{2}I\omega^2$. However, the rod is rotating about its *endpoint*, not its center, so we have the wrong moment of inertia. Rotating about an endpoint is just rotating about a parallel axis a distance $L/2$ from the center, so we can easily find the moment of inertia about the endpoint from the given one for rotation about the center:

$$I_{\text{end}} = I_{\text{center}} + M\left(L/2\right)^2 = \frac{1}{2}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2 \tag{6}$$

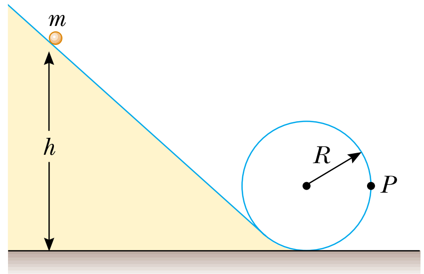
What is left is just to relate kinetic and potential energy:

$$Mg\left(\frac{L}{2}\right) = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2 \tag{7}$$

$$\Longrightarrow \qquad \omega = \frac{3g}{L} \tag{8}$$

Could you have used torque? Sure. This would have found you the angular acceleration $\alpha = d\omega/dt$, from which you could find ω. *However*, the angular acceleration is not constant as the rod is falling, as it will explicitly depend on the angle the rod makes with the vertical! This means you can't use the simple equations we normally use for constant acceleration, and matters are more complicated. Best to use energy when you can get away with it.

3. A solid sphere of mass m and radius r rolls without slipping along the track shown below. It starts from rest with the lowest point of the sphere at a height h above the bottom of the loop of radius R, much larger than r. What is the minimum value of h (in terms of R) such that the sphere completes the loop? Do not ignore the rotational kinetic energy ... The moment of inertia for a solid sphere is $I = \frac{2}{5}mr^2$.



Solution: Conservation of energy will allow us to find the velocity at the top of the loop, and we can find the minimum velocity required to stay on the track by considering the forces at the top of the loop. Comparing the two will give us an expression for h in terms of R. In fact, we've already done this problem for point masses, all we really need to do differently is keep track of rotational kinetic energy. First, conservation of energy.

Let the ground level be our zero point for potential energy. Before the mass is released, it has only potential energy based on its height h above the ground. At the top of the loop, it still has potential energy due to its height $2R$ above the ground, but now also has linear kinetic energy due to the motion of its center of mass at speed v_{cm} and rotational kinetic energy due to its rotating at angular velocity ω.

$$mgh = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\omega^2 + mg\,(2R) \tag{9}$$

$$mg\,(h - 2R) = \frac{1}{2}m\left(v_{cm}^2 + I\omega^2\right) \tag{10}$$

We can relate v_{cm} and ω by noting that the horizontal distance the sphere covers in rolling through θ radians is the arclength of the circle through the same angle, $r\theta$, and the angle θ at constant angular velocity is ωt. Since the horizontal distance covered is also $v_{cm}t$, we have $r\omega t = v_{cm}t$, or $\omega = v_{cm}/r$.

$$mg\,(h - 2R) = \frac{1}{2}m\left(v_{cm}^2 + I\frac{v^2}{r^2}\right) \tag{11}$$

$$2g\,(h - 2R) = v^2\left(1 + \frac{I}{mr^2}\right) \tag{12}$$

$$v^2 = \frac{2g\,(h - 2R)}{1 + I/mr^2} \tag{13}$$

This is the actual speed the sphere will have at the top of the loop. We must compare this to the minimum speed required by centripetal acceleration. At the top of the loop, the only two forces will be the sphere's weight mg and the normal force F_n, both pointing downward toward the center of the circle. These two forces must equal the centripetal force required to stay on the track, which also acts downward toward the center of the circle:

$$\sum F = mg + F_n = \frac{mv^2}{R} \tag{14}$$

$$F_n = \frac{mv^2}{R} - mg \tag{15}$$

The sphere will stay on the track so long as the normal force is positive, i.e., when

$$v^2 > Rg \tag{16}$$

The actual speed of the sphere must be larger than this. Using the speed we found, we can solve for h to find the minimum requisite height.

$$v^2 = \frac{2g\,(h - 2R)}{1 + I/mr^2} > Rg \tag{17}$$

$$2g\,(h - 2R) > Rg\left(1 + \frac{I}{mr^2}\right) \tag{18}$$

$$h - 2R > \frac{R}{2}\left(1 + \frac{I}{mr^2}\right) \tag{19}$$

$$h > R\left(2 + \frac{1}{2} + \frac{I}{2mr^2}\right) \tag{20}$$

Noting that $I = \frac{2}{5}mr^2$, $I/2mr^2 = \frac{1}{5}$, so

$$h > R\left(2 + \frac{1}{2} + \frac{1}{5}\right) = \frac{27}{10}R = 2.7R \tag{21}$$