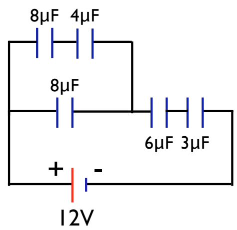
UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 106 / LeClair Spring 2008

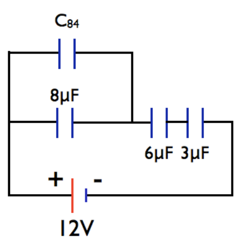
Exercise: Electrical Energy & Capacitance: Solutions

1. Find the equivalent capacitance of the capacitors in the figure below.



Before we start, it is useful to remember that one farad times one volt gives one coulomb: $1\,[\text{F}]\cdot 1\,[\text{V}] = 1\,[\text{C}]$, and that capacitance times voltage gives stored charge: $Q = CV$. Knowing this now will save some confusion on units later on. For that matter, it is also good to remember that the prefix μ means 10^{-6}.

In order to find a single equivalent capacitor that could replace all five in the diagram above, we need to look for purely series and parallel combinations that can be replaced by a single capacitor. The uppermost $8\,\mu$F and $4\,\mu$F capacitors are purely in series, so they can be replaced by a single equivalent we will call C_{84}, as shown below:

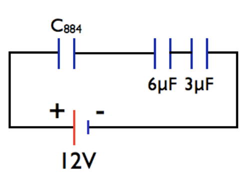


Using our rule for combining series capacitors, we can find the value of C_{84} easily:

$$\frac{1}{C_{84}} = \frac{1}{8\,\mu\text{F}} + \frac{1}{8\,\mu\text{F}}$$

$$\Longrightarrow \quad C_{84} = \frac{8}{3}\,\mu\text{F} \approx 2.67\,\mu\text{F}$$

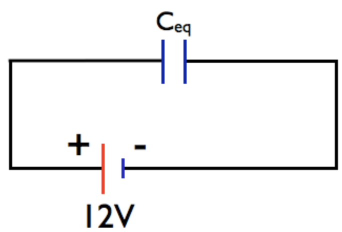
Now we have this equivalent capacitance purely in *parallel* with the second $8\,\mu$F capacitor. We can replace C_{84} and the second $8\,\mu$F capacitors with a single equivalent, which we will call C_{884}:



Using our addition rule for parallel capacitors, we can find its value:

$C_{884} = C_{84} + 8\,\mu\text{F} \approx 10.67\,\mu\text{F}$

This leaves us with three capacitors in series, as shown below:



Adding together these three in series, we have the overall equivalent capacitance, C_{eq}:

$$\frac{1}{C_{eq}} = \frac{1}{C_{884}} + \frac{1}{3\,\mu\text{F}} + \frac{1}{6\,\mu\text{F}}$$
$$\implies \quad C_{eq} \approx 1.68\,\mu\text{F}$$

Since we now have one single capacitor connected to a single voltage source, we can find the total charge stored in the equivalent capacitor, Q_{eq}:

$$Q_{eq} = C_{eq}V = (1.68\,\mu\text{F})\,(12\,\text{V}) \approx 20.16\,\mu\text{C}$$

Now what if we wanted to get the charge and voltage on each single capacitor? In that case, we have to work backwards and rebuild our original circuit. We know that the C_{eq} capacitor is really three capacitors in series - the $6\,\mu$F, the $3\,\mu$F, and C_{884}. Series capacitors always have the same charge, and one must have the same charge as the equivalent capacitor: $Q_{6\mu\text{F}} = Q_{3\mu\text{F}} = Q_{884}$ Since we know the charge and capacitance for all three of these capacitors, we can now find the voltage on each, since $V = Q/C$:

$$V_{6\mu\text{F}} = \frac{Q_{6\mu\text{F}}}{6\mu\text{F}} = \frac{20.16\,\mu\text{C}}{6\mu\text{F}} \approx 3.4\,\text{V}$$
$$V_{3\mu\text{F}} = \frac{Q_{3\mu\text{F}}}{3\mu\text{F}} = \frac{20.16\,\mu\text{C}}{3\mu\text{F}} \approx 6.7\,\text{V}$$
$$V_{884} = \frac{Q_{884}}{C_{884}} = \frac{20.16\,\mu\text{C}}{10.67\,\mu\text{F}} \approx 1.9\,\text{V}$$

Notice that the voltage on all three of these series capacitors adds up to the total battery voltage - it must be so, based on conservation of energy. Next, we know that C_{884} is really two capacitors in parallel - the lower $8\,\mu$F capacitor and C_{84}. Parallel capacitors have the same voltage, so we know that both of these have to have $V_{\text{lower }8\mu\text{F}} = V_{84} = V_{884} = 1.9\,\text{V}$ across them. We know the voltage and the capacitance for C_{84} and the lower $8\,\mu$F capacitors now, so we can find the stored charge, $Q = CV$:

$$Q_{\text{lower }8\mu\text{F}} = 8\,\mu\text{F} \cdot V_{884} = 8\,\mu\text{F} \cdot 1.9\,\text{V} \approx 15.2\,\mu\text{C}$$
$$Q_{84} = C_{84} \cdot V_{884} = 2.67\,\mu\text{F} \cdot 1.9\,\text{V} \approx 5.1\,\mu\text{C}$$

Finally, the capacitor C_{84} is really two capacitors in series, which must both have the same charge: $Q_{4\mu\text{F}} = Q_{\text{upper }8\mu\text{F}} = Q_{84}$. Given the charge on both of the remaining capacitors and their capacitances, we can find the voltages:

$$V_{4\mu\text{F}} = \frac{Q_{4\mu\text{F}}}{4\mu\text{F}} = \frac{5.1\,\mu\text{C}}{4\mu\text{F}} \approx 1.26\,\text{V}$$

$$V_{\text{upper }8\mu\text{F}} = \frac{Q_{\text{upper }8\mu\text{F}}}{8\mu\text{F}} = \frac{5.1\,\mu\text{C}}{8\mu\text{F}} \approx 0.63\,\text{V}$$

Now we know the charge and voltage on every single capacitance, as well as the overall charge (Q_{eq}) and effective capacitance (C_{eq}). Your numbers may be very slightly different than those above due to different choices in rounding, this is normal. The results are summarized in the table below:

Table 1: *Equivalent capacitances, charges, and voltages*

Capacitor [µF]	**Charge [µC]**	**Voltage [V]**
top 8 μF	5.1	0.63
4 μF	5.1	1.26
lower 8 μF	15.2	1.9
6 μF	20.16	3.4
3 μF	20.16	6.7
C_{eq}	20.16	12

2. A parallel-plate capacitor has $4.00\,\text{cm}^2$ plates separated by $6.00\,\text{mm}$ of air. If a $12.0\,\text{V}$ battery is connected to this capacitor, how much energy does it store in Joules? In electron volts?

If we want to find the energy stored in the capacitor, we need to know two of three things, minimally: the amount of charge stored, the voltage applied, and the capacitance. Any two of these three are sufficient, based on our formula for the potential energy stored in a capacitor:

$$\Delta PE = \frac{1}{2}Q\Delta V = \frac{1}{2}C\left(\Delta V\right)^2 = \frac{Q^2}{2C}$$

We already know the applied voltage, $\Delta V = 12.0\,\text{Volts}$. Since this is a parallel plate capacitor and we know its area A and plate spacing d we can easily calculate the capacitance ... if we are very careful with units. Recall that the dielectric constant of air is essentially one ($\kappa \approx 1$).

$$\begin{aligned} C &= \frac{\kappa\epsilon_0 A}{d} \\ &= \frac{1 \cdot \epsilon_0 \left(4.00\,\text{cm}^2\right) \cdot \left(\frac{1\,\text{m}}{100\,\text{cm}}\right)^2}{6.00 \times 10^{-3}\,\text{m}} \\ &= \frac{\left(8.85 \times 10^{-12}\,\text{F/m}\right) \cdot \left(4 \times 10^{-4}\,\text{m}^2\right)}{6.00 \times 10^{-3}\,\text{m}} \\ &= \frac{8.85 \times 4.00}{6.00} \cdot 10^{-13}\,\text{F} \\ &\approx 5.90 \times 10^{-13}\,\text{F} = 0.590\,\text{pF} \end{aligned}$$

Now we know the capacitance and the voltage, we can find the energy readily:

$$\Delta PE = \frac{1}{2}\left(5.90\times10^{-13}\,\text{F}\right)(12.0\,\text{V})^2 = 4.25\times10^{-11}\,\text{F}\cdot\text{V}^2 = 4.25\times10^{-11}\,\text{J}$$
$$= \left(4.25\times10^{-11}\,\text{J}\right)\left(\frac{1\,\text{eV}}{1.60\times10^{-19}\,\text{J}}\right) = 2.66\times10^{8}\,\text{eV} = 266\,\text{MeV}$$

This problem brings to mind a few handy SI unit conversions, which you should be able to verify: $1\,\text{J}=1\,\text{F}\cdot1\,\text{V}^2=1\,\text{C}\cdot1\,\text{V}$, $1\,\text{C}=1\,\text{F}\cdot1\,\text{V}$.

3. A potential difference of 100 mV exists between the outer and inner surfaces of a cell membrane. The inner surface is negative relative to the outer. How much work is required to move a sodium ion Na^+ outside the cell from the interior? Answer in electron volts and Joules. A singly-charged ion has a charge of $1e$, $1\,\text{eV}=1.6\times10^{-19}\,\text{J}$.

The work done in moving a charge q across a potential difference ΔV is readily calculated: $W=-\Delta PE=-q\Delta V$. In this case the charge is $e=1.6\times10^{-19}\,\text{C}$, and $\Delta V=0.1\,\text{V}$. Watch how easy it is to find the answer in electron volts:

$$W = -q\Delta V = \left(1.6\times10^{-19}\,\text{C}\right)(0.1\,\text{V})\cdot\left(\frac{1\,\text{eV}}{1.6\times10^{-19}\,\text{J}}\right)$$
$$= -\left(\cancel{1.6\times10^{-19}}\,\text{C}\right)(0.1\,\text{V})\cdot\left(\frac{1\,\text{eV}}{\cancel{1.6\times10^{-19}}\,\text{J}}\right)$$
$$= -0.1\,\frac{\cancel{[\text{C}\cdot\text{V}]}\cdot\text{eV}}{\cancel{\text{J}}} \qquad \textit{remember: } 1\,\text{J}=1\,\text{C}\cdot1\,\text{V}$$
$$= -0.1\,\text{eV}$$

In fact, we didn't even need to go through all that. An electron volt is *defined* as the energy required to move one electron's equivalent of charge - $1e$ - through a potential difference of 1 Volt. Our ion has the same magnitude of charge as an electron, and we move it through 0.1 V. Following the definition of an electron volt, we must have $\Delta PE=0.1\,\text{eV}$. This is one reason why electron volts are such a handy unit for many areas of physics.

Anyway: how about the answer in Joules? Well, if $1\,\text{eV}=1.6\times10^{-19}\,\text{J}$, then $0.1\,\text{eV}=1.6\times10^{-20}\,\text{J}$.

4. A point charge q is a distance x above an infinite conducting plate. Given that the electric field above the plate must be $4\pi k_e\sigma$, calculate the surface charge density as a function of the position on the plate.

As we discussed in class, a conductor acts as a mirror for electric field lines, which allows us to replace some difficult problems involving conductors with equivalent problems involving point charges. The simplest example of this is shown below: a point charge just above an infinite conducting plate.

Our real problem involves a point charge $+q$ a distance x from a conducting sheet. Since the conducting sheet acts like a mirror for electric field lines, we can replace the conducting plane by a second point charge $-q$ at a distance $2x$ from the original charge, or a dipole. This virtual charge will give *exactly* the same electric field as the induced charges in the conducting plate will - the field lines from $+q$ have to intersect the conducting plate perpendicular to its surface, which is exactly what the field of our dipole looks like. We have already solved for the field from a dipole completely, so in fact we have solved this problem too, and we know what the field is anywhere we choose to specify.

We solved the *virtual* problem which gives the same answer as our problem. What about the *real* charges induced in the plate? We know our positive point charge will attract negative charges within the plate and draw them toward it, building up a local region of negative charge directly below it. Since we know what the electric field is anywhere on the plate -

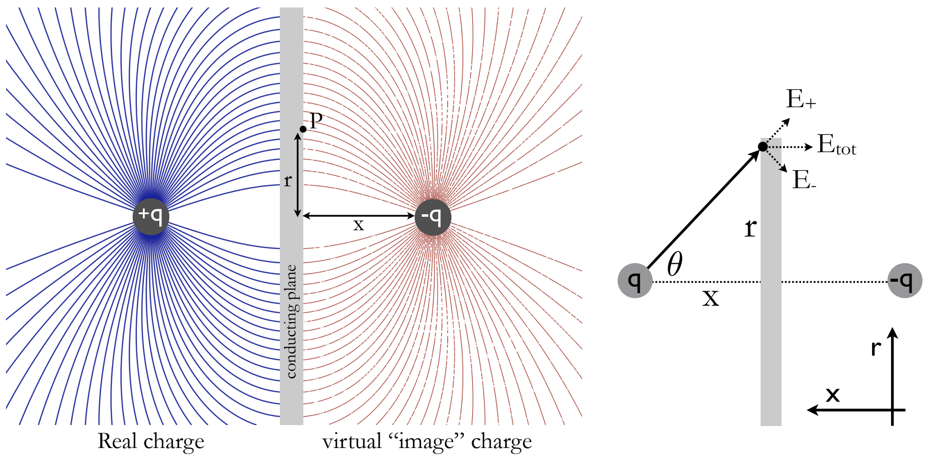


Figure 1: Left: *The field of a charge near a conducting plane, found by the method of images.* **Right:** *Setup for calculating the field.*

it *must* be the same as that of a dipole along its central dividing plane - we can also find the precise distribution of this charge.

On very general grounds, we already derived that above an infinite conducting plane, the electric field must have a value $4\pi k_e\sigma$, where σ is the surface charge density. Thus, all we have to do is find the electric field at an arbitrary position along the plate, and we will have the surface charge density. We will call the distance from the positive charge to the plate x, meaning the image charge is a distance x below the plate, and the lateral position along the plate itself will be r.[i] The electric field due to the real $+q$ charge is easily found:

$$|\vec{\mathbf{E}}|_+ = \frac{k_e q}{r^2+x^2}$$

Clearly, the field from the virtual $-q$ charge E_- is the same in magnitude. Since both fields must be along a line connecting the point of interest on the plate with their respective charges, the symmetry of the system dictates that the vertical components of the two fields must cancel, leaving us with a net field in the $-x$ direction. The total field is then the sum of the horizontal components of E_+ and E_-, which are identical. We should not forget the $-$ sign, since we are defining the $+q$ charge to be a positive distance x above the plate.

$$|\vec{\mathbf{E}}_{tot}| = E_{+,x} + E_{-,x} = 2E_{+,x} = 2|\vec{\mathbf{E}}_+|\cos\theta = \frac{-2k_e q}{r^2+x^2}\cos\theta = \frac{2k_e q x}{\left(r^2+x^2\right)^{3/2}}$$

In the last step, we used the relationship $\cos\theta = x/\sqrt{r^2+x^2}$. As required by our boundary conditions for $\vec{\mathbf{E}}$, the field is perpendicular to the plate. This total field must also be $4\pi k_e\sigma$, thus:

$$\sigma(r) = \frac{1}{4\pi k_e}\frac{-2k_e q x}{\left(r^2+x^2\right)^{3/2}} = \frac{-qx}{2\pi\left(r^2+x^2\right)^{3/2}}$$

And that is it, the surface charge density as a function of the lateral position on the plate r and the distance of the $+q$

[i] Which means we are really setting up an (r, θ, x) cylindrical coordinate system, if you are interested in such things.

charge from the plate x. The surface charge density is really only a function of r, as it exists only on the surface of the plate; x is in this case basically a parameter that characterizes the strength of the polarizing $+q$ charge. For an actual dipole, rather than a virtual one, the product of the charge and separation ($2qx$ in this case) is known as the *dipole moment*, and it really is a (vector) measure of a dipole's effective strength.

This result makes some sense: in the plane of the plate the charge density must be radially symmetric, and this is borne out by the lack of θ dependence in our final result. Below, we plot a cross-section of the surface charge density along the plate for various values of x.

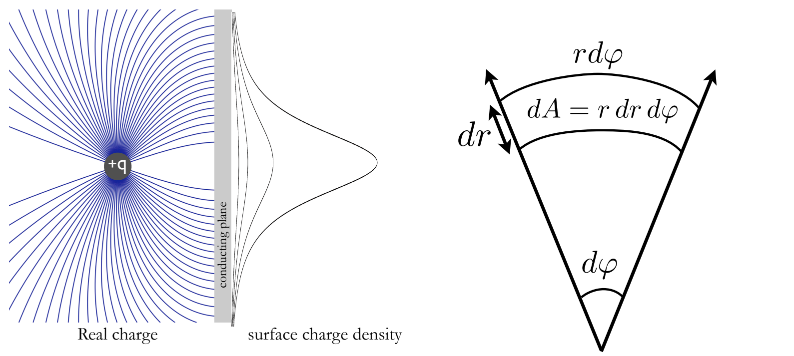


Figure 2: Left: *Surface charge density as a function of the lateral position on the plate for various distances from the plate.* **Right:** *Element of area in polar coordinates.*

There is of course one more check we can make. If our method of images is correct, then all of the surface charge density over the entire plate must just add up to $-q$, the same as the image charge we placed.[ii] Within the conducting plane, we can integrate σdA over the whole plate to find the total charge. Since the charge density is radially symmetric, it makes sense to integrate over area in polar coordinates - let the radial distance vary from zero to infinity, and sweep the radius through an in-plane angle φ.

An element of area in polar coordinates is $r\,dr\,d\theta$, you should be able to see how this comes about from the figure above. We will integrate over all possible patches like this on the sheet, which means taking r from $0 \to \infty$, and φ from $0 \to 2\pi$. The integration over φ is trivial, since nothing depends on the angle, and it just adds a factor 2π.

$$q_{\text{plate}} = \int \sigma dA = \int_0^{2\pi} d\varphi \int_0^{\infty} \sigma r\,dr = \int_0^{2\pi} d\varphi \int_0^{\infty} \frac{-qxr}{2\pi\left(r^2+x^2\right)^{3/2}}\,dr = -q\int_0^{\infty} \frac{xr}{\left(r^2+x^2\right)^{3/2}}\,dr$$

$$= -q\left[-\frac{x}{\sqrt{r^2+x^2}}\right]_0^{\infty} = q\left[\frac{x}{\sqrt{r^2+x^2}}\right] = q\,[0-1]_0^{\infty} = -q$$

No problem.

[ii] There are deep and complicated theorems that prove that this *must* be true. We will not worry about the general case, but just prove our one solution is sensible.