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UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 106-4 / LeClair — Fall 2008

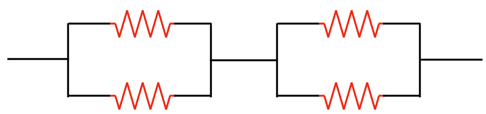
Circuits Exercises

1. Are the two headlights of a car wired in series or in parallel? How can you tell?

Have you ever seen cars driving down the road with only one working headlight? If headlights were wired in series, when one light goes out, both would go out. Wiring headlights in parallel means that when one bulb goes out, the other stays lit.

2. What advantage might there be in using two identical resistors in parallel connected in series with another identical parallel pair, rather than just using a single resistor?

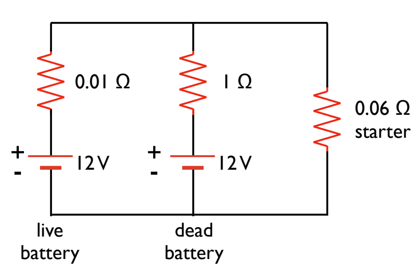
The combination we are talking about is this one:



You can verify for yourself that if each individual resistor has a value R, the equivalent resistance of the arrangement above is also R. The advantage in this situation compared to using a single resistor of value R is that while the *total* power dissipation is the same, it is now divided between four resistors. This arrangement lets one use several physically smaller low power components instead of one bulky high-power component. For instance, if you only had resistors rated at 15 W, but your circuit required 30 W, one could use the arrangement above safely.

Each in this case would each have half of the total voltage (due to the series combination) and half the total current (due to the parallel combination). Since power is current times voltage, the total power in any given resistor is one quarter what it would be for a single resistor connected to the same power source.
Another advantage would be redundancy, as with the headlights in question 1 - in this arrangement, one single failure will still allow the circuit to operate. For instance, the four resistors might be electric heaters connected to a constant voltage source (like a wall socket). A single heater could fail, and if the rest of the circuit were properly designed, the remaining three would still provide 2/3 of the original power.

3. A dead battery is charged by connecting it to the live battery of another car with jumper cables (see below). Determine the current in the starter and in the dead battery.



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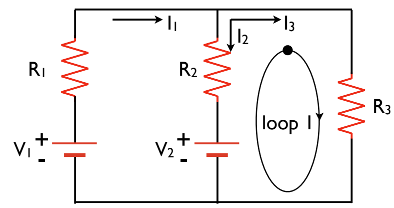
Since this circuit has several branches and multiple batteries, we cannot reduce it by using our rules of series and parallel resistors - we have to use Kirchhoff's rules. In order to do that, we first need to assign currents in each branch of the circuit. It doesn't matter what directions we choose at all, assigning directions is just to define what is, relatively speaking, positive and negative. If we choose the direction for one current incorrectly, we will get a negative number for that current to let us know. Below, we choose initial currents I_1, I_2, and I_3 in each branch of the circuit.



loop 2 = outer perimeter, CW

Here we have also labeled each component symbolically to make the algebra a bit easier to sort out. Note that since we have three unknowns - the three currents - so we will need three equations to solve this problem completely.

Now we are ready to apply the rules. First, the junction rule. We have only two junctions in this circuit, in the center at the top and bottom where three wires meet. The junction rule basically states that the current into a junction (or node) must equal the current out. In the case of the upper node, this means:

$$I_1 = I_2 + I_3 \tag{1}$$

You can easily verify that the lower node gives you the same equation. Next, we can apply the loop rule. There are three possible loops we can take: the rightmost one containing R_3 and R_2, the leftmost one containing R_1 and R_2, and the outer perimeter (containing R_1 and R_3). We only need to work through two of them - we have already one equation above, and we only need two more. Somewhat arbitrarily, we will pick the right side and perimeter loops.

First, the outer loop. Start just above the live battery V_1, and walk *clockwise* around the loop. We cross the battery from positive to negative for a *gain* in potential energy, and we cross R_1 and R_3 in the direction of current flow for a *loss* of potential energy. These three have to sum to zero for a closed loop:

$$V_1 - I_1R_1 - I_3R_3 = 0 \tag{2}$$

Next, the right-hand side loop. Again, start just above the battery (V_2 this time), and walk *clockwise* around the loop. Now we cross the battery and R_3 for a gain and loss of voltage, respectively, but then cross R_2 in the opposite direction of the current - this gives a voltage *gain*:

$$V_2 - I_3R_3 + I_2R_2 = 0 \tag{3}$$

Now we have three equations and three unknowns, and we are left with the pesky problem of solving them for the three currents. There are many ways to do this, we will illustrate two of them. Before we get started, let us repeat the three questions in a more symmetric form.

$$\begin{aligned}
I_1 - I_2 - I_3 &= 0 \\
R_1I_1 + R_3I_3 &= V_1 \\
R_2I_2 - R_3I_3 &= -V_2
\end{aligned}$$

The first way we can proceed is by substituting the first equation into the second:

$$\begin{aligned}
&V_1 = R_1I_1 + R_3I_3 = R_1\left(I_2 + I_3\right) + R_3I_3 = R_1I_2 + \left(R_1 + R_3\right)I_3 \\
\Longrightarrow \quad &V_1 = R_1I_2 + \left(R_1 + R_3\right)I_3
\end{aligned}$$

Now our three equations look like this:

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\begin{aligned}
I_1 - I_2 - I_3 &= 0 \\
R_1 I_2 + (R_1 + R_3)\, I_3 &= V_1 \\
R_2 I_2 - R_3 I_3 &= -V_2
\end{aligned}
$$

The last two equations now contain only I_1 and I_2, so we can solve the third equation for I_2 ...

$$I_2 = \frac{I_3 R_3 - V_2}{R_2}$$

... and plug it in to the second one:

$$
\begin{aligned}
V_1 &= R_1 I_2 + (R_1 + R_3)\, I_3 = R_1 \left(\frac{I_3 R_3 - V_2}{R_2}\right) + (R_1 + R_3)\, I_3 \\
V_1 &+ \frac{V_2 R_1}{R_2} - \left(R_1 + R_3 + \frac{R_1 R_3}{R_2}\right) I_3 = 0 \\
I_3 &= \frac{V_1 + \frac{V_2 R_1}{R_2}}{R_1 + R_3 + \frac{R_1 R_3}{R_2}} \\
I_3 &= \frac{V_1 R_2 + V_2 R_1}{R_1 R_2 + R_2 R_3 + R_1 R_3} \approx 172\,\mathrm{A}
\end{aligned}
$$

Now that you know I_3, you can plug it in the expression for I_2 above, you should find $I_2 \approx -1.7\,\mathrm{A}$. Why negative? All that means is that our original guess for the direction of I_2 was wrong - rather than flowing down the center wire, it actually flows up.[i]

What is the second way to solve this? We can start with our original equations, but in a different order:

$$
\begin{aligned}
I_1 - I_2 - I_3 &= 0 \\
R_2 I_2 - R_3 I_3 &= -V_2 \\
R_1 I_1 + R_3 I_3 &= V_1
\end{aligned}
$$

The trick we want to use is formally known as 'Gaussian elimination,' but it just involves adding these three equations together in different ways to eliminate terms. First, take the first equation above, multiply it by $-R_1$, and add it to the third:

$$
\begin{aligned}
& [-R_1 I_1 + R_1 I_2 + R_1 I_3] = 0 \\
& \underline{+ \qquad\qquad R_1 I_1 + R_3 I_3 = V_1} \\
\Longrightarrow \quad & R_1 I_2 + (R_1 + R_3)\, I_3 = V_1
\end{aligned}
$$

Now take the second equation, multiply it by $-R_1/R_2$, and add it to the new equation above:

$$
\begin{aligned}
& -\frac{R_1}{R_2}\left[R_2 I_2 - R_3 I_3\right] = -\frac{R_1}{R_2}\left[-V_2\right] \\
& \underline{+ \qquad\qquad R_1 I_2 + (R_1 + R_3)\, I_3 = V_1} \\
\Longrightarrow \quad & \left(\frac{R_1 R_3}{R_2} + R_1 + R_3\right) I_3 = \frac{R_1}{R_2} V_2 + V_1
\end{aligned}
$$

Now the resulting equation has only I_3 in it. Solve this for I_3, and proceed as above.

Optional: There is one more way to solve this set of equations using matrices and Cramer's rule,[ii] if you are familiar with this technique. If you are not familiar with matrices, you can skip to the next problem - you are not required or necessarily expected to know how to do this. First, write the three equations in matrix form:

[i] If you think about that, it means that we aren't charging the battery at all, but still draining it. Hopefully the 171 A through the starter is enough to turn over the engine.

[ii] See 'Cramer's rule' in the Wikipedia to see how this works.

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$$\begin{bmatrix} R_1 & 0 & R_3 \\ 0 & R_2 & -R_3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \\ 0 \end{bmatrix}$$
$$\mathbf{aI} = \mathbf{V}$$

The matrix $\mathbf{a}$ times the column vector $\mathbf{I}$ gives the column vector $\mathbf{V}$, and we can use the determinant of the matrix $\mathbf{a}$ with Cramer's rule to find the currents. For each current, we construct a new matrix, which is the same as the matrix $\mathbf{a}$ except that the the corresponding column is replaced the column vector $\mathbf{V}$. Thus, for I_1, we replace column 1 in $\mathbf{a}$ with $\mathbf{V}$, and for I_2, we replace column 2 in $\mathbf{a}$ with $\mathbf{V}$. We find the current then by taking the new matrix, calculating its determinant, and dividing that by the determinant of $\mathbf{a}$. Below, we have highlighted the columns in $\mathbf{a}$ which have been replaced to make this more clear:

$$I_1 = \frac{\begin{vmatrix} V_1 & 0 & R_3 \\ -V_2 & R_2 & -R_3 \\ 0 & -1 & -1 \end{vmatrix}}{\det \mathbf{a}} \qquad I_2 = \frac{\begin{vmatrix} R_1 & V_1 & R_3 \\ 0 & -V_2 & -R_3 \\ 1 & 0 & -1 \end{vmatrix}}{\det \mathbf{a}} \qquad I_3 = \frac{\begin{vmatrix} R_1 & 0 & V_1 \\ 0 & R_2 & -V_2 \\ 1 & -1 & 0 \end{vmatrix}}{\det \mathbf{a}}$$

Now we need to calculate the determinant of each new matrix, and divide that by the determinant of $\mathbf{a}$.[iii] First, the determinant of $\mathbf{a}$.

$$\det a = -R_1R_2 - R_1R_3 + 0 - 0 + 0 - R_2R_3 = -\left(R_1R_2 + R_2R_3 + R_1R_3\right)$$

We can now find the currents readily from the determinants of the modified matrices above and that of $\mathbf{a}$ we just found:

$$I_1 = \frac{-V_1R_2 - V_1R_3 + 0 - 0 + V_2R_3 - 0}{-\left(R_1R_2 + R_2R_3 + R_1R_3\right)} = \frac{V_1\left(R_2 + R_3\right) - V_2R_3}{R_1R_2 + R_2R_3 + R_1R_3} \approx 170\,\text{A}$$

$$I_2 = \frac{R_1V_2 - 0 - V_1R_3 - 0 + 0 + R_3V_2}{-\left(R_1R_2 + R_2R_3 + R_1R_3\right)} = \frac{R_3V_1 - V_2\left(R_1 + R_3\right)}{R_1R_2 + R_2R_3 + R_1R_3} \approx -1.7\,\text{A}$$

$$I_3 = \frac{0 - R_1V_2 + 0 - 0 + 0 - V_1R_2}{-\left(R_1R_2 + R_2R_3 + R_1R_3\right)} = \frac{R_1V_2 + R_2V_1}{R_1R_2 + R_2R_3 + R_1R_3} \approx 171.6\,\text{A}$$

These are the same results you would get by continuing on with either of the two previous methods. Both numerically and symbolically, we can see from the above that $I_1 = I_2 + I_3$:

$$I_2 + I_3 = \frac{R_3V_1 - V_2\left(R_1 + R_3\right) + R_1V_2 + R_2V_1}{R_1R_2 + R_2R_3 + R_1R_3} = \frac{V_1\left(R_2 + R_2\right) + V_2\left(R_1 - R_1 - R_3\right)}{R_1R_2 + R_2R_3 + R_1R_3} = \frac{V_1\left(R_2 + R_2\right) - V_2R_3}{R_1R_2 + R_2R_3 + R_1R_3} = I_1$$

4. Two resistors connected in series have an equivalent resistance of $690\,\Omega$. When they are connected in parallel, their equivalent resistance is $150\,\Omega$. Find the resistance of each resistor.

When we combine two resistors in series, they simply add to form a equivalent resistor. In parallel, they add inversely. This implies two equations:

$$R_1 + R_2 = 690$$
$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{150}$$

It is more convenient if we rearrange the second one (find a common denominator for the left-hand side and invert) to look like this:

$$\frac{R_1R_2}{R_1 + R_2} = 150$$

Now, plug the first one in to the second and massage it a bit:

$$\frac{R_1R_2}{R_1 + R_2} = \frac{R_1R_2}{690} = 150$$
$$R_1R_2 = 150 \cdot 690$$

We can use our first equation a second time, noting that $R_2 = 690 - R_1$:

[iii] Again, the Wikipedia entry for 'determinant' is quite instructive.

$$
\begin{aligned}
R_1 R_2 = R_1 \left(690 - R_1\right) = 690R_1 - R_1^2 &= 150 \cdot 690 \\
\implies \quad R_1^2 - 690R_1 + 150 \cdot 690 &= 0
\end{aligned}
$$

Now we have a quadratic that we can solve for R_1.

$$
\begin{aligned}
R_1 &= \frac{-\left(-690\right) \pm \sqrt{\left(-690\right)^2 - 4 \cdot 1 \cdot \left(150 \cdot 690\right)}}{2} \\
&= \frac{690 \pm \sqrt{690^2 - 600 \cdot 690}}{2} \\
&= \frac{690 \pm 690\sqrt{1 - \frac{600}{690}}}{2} \\
&= 345 \left[1 \pm \sqrt{1 - \frac{600}{690}}\right] \\
&= 345 \left[1 \pm \sqrt{1 - \frac{20}{23}}\right] \\
&\approx 220.4, 469.6
\end{aligned}
$$

Now we have two solutions for R_1. What is that? No worries. Since we labeled R_1 and R_2 arbitrarily, and our equations are completely symmetric with regard to either, we have actually just found *both* R_1 and R_2. Try plugging them both in to the first equation, and you will see that we really only have one complete solution:

$$
\begin{aligned}
R_2 &= 690 - R_1 = 690 - 220.4 = 496.6 \qquad \text{1st solution} \\
R_2 &= 690 - R_1 = 690 - 469.6 = 220.4 \qquad \text{2nd solution}
\end{aligned}
$$

Thus, our two resistors have to be $496.6\,\Omega$ and $220.4\,\Omega$.