

UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 106-4 / LeClair

Fall 2008

Problem Set 3 Solutions

The solutions below are generally meant to be more precise and nitpicking than I am expecting you to be, and I have tried in some cases to point out multiple ways to approach the same problem. In some cases this means more advanced solutions that I don't necessarily expect you to come up with, but that some of you might find enlightening. You should not feel like you needed to have everything here to receive full credit, and I certainly have not outlined *every* correct way to solve the problems ... the idea is that even if you found correct solutions for all the problems, you will probably still learn something.

1. *Griffiths 2.8, 2.32* A solid sphere of radius R has a uniform charge density ρ and total charge Q . Derive an expression for its total electric potential energy. (Suggestion: imagine that the sphere is constructed by adding successive layers of concentric shells of charge $dq = (4\pi r^2 dr)\rho$ and use $dU = V dq$.)

There are in my opinion three reasonably straightforward ways to go about this problem, each with its own merits. Below I will outline all three, you only needed to have one valid solution for full credit.

Method 1: use $dU = V dq$

We know that we can find the potential energy for any charge distribution by calculating the work required to bring in each bit of charge in the distribution in from an infinite distance away. If we bring a bit of charge dq in from infinitely far away (where the electric potential is zero) to a point $P(\vec{r})$, at which the electric potential is V , work is required. The potential V results from all of the other bits of charge already present; we bring in one dq at a time, and as we build up the charge distribution, we calculate the new potential, and that new potential determines the work required to bring in the next dq .

In the present case, we will build our sphere out of a collection of spherical shells of infinitesimal thickness dr . We'll start with a shell of radius 0, and work our way up to the last one of radius R . A given shell of radius r will have a thickness dr , which gives it a surface area of $4\pi r^2$ and a volume of (thickness)(surface area) = $4\pi r^2 dr$.

Once we've built the sphere up to a radius r , Gauss' law tells us that the potential at the surface is just that of a point charge of radius r :

$$V(r) = \frac{k_e q(r)}{r}$$

Where $q(r)$ is the charge built up so far, contained in a radius r . Bringing in the next spherical shell of radius $r + dr$ and charge dq will then require work to be done, in the amount $V(r) dq$, since we are bringing a charge dq from a potential of 0 at an infinite distance to a potential $V(r)$ at a distance r . Thus, the little bit of work required to bring in the next shell is

$$-dW = dU = V(r) dq = \frac{k_e q(r)}{r} dq$$

Now, first we'll want to know what $q(r)$ is. If we have built the sphere to a radius r , then the charge contained so far is just the charge density times the volume of a sphere of radius r :

$$q(r) = \frac{4}{3}\pi r^3 \rho$$

Next, we need to know what dq is, the charge contained in the next shell of charge we want to bring in. In this case the charge is just the volume of the shell times the charge density:

$$dq = 4\pi r^2 dr \rho$$

Putting that all together:

$$dU = \frac{k_e \frac{4}{3}\pi r^3 \rho}{r} 4\pi r^2 dr \rho = \frac{16\pi^2 k_e \rho^2}{3} r^4 dr$$

All we need to do now is integrate up all the small bits of work required from the starting radius $r=0$ to the final radius $r=R$:

$$U = \int_0^R dU = \int_0^R \frac{16\pi^2 k_e \rho^2}{3} r^4 dr = \frac{16\pi^2 k_e \rho^2}{3} \left[\frac{r^5}{5} \right]_0^R$$

$$\implies U = \frac{16\pi^2 k_e \rho^2}{15} R^5$$

If we remember that the total charge on the sphere is $Q_{\text{tot}} = \frac{4}{3}\pi R^3 \rho$, we can rewrite a bit more simply:

$$U = \frac{3k_e Q_{\text{tot}}^2}{5R}$$

Method 2: use $dU = \frac{1}{2}\epsilon_o E^2 dV_{ol}$

We have also learned that the potential energy of a charge distribution can be found by integrating its electric field (squared) over all space:ⁱ

$$U = \frac{1}{2} \int_{\text{everywhere}} \epsilon_o E^2 dV_{ol} = \frac{1}{8\pi k_e} \int_{\text{everywhere}} E^2 dV_{ol}$$

If we have a sphere of charge of radius R , we can break all of space up into two regions: the space outside of the sphere, where the field is $\vec{E}_{\text{out}}(r)$ at a distance r from the center of the sphere, and the space inside the sphere, where the field is $\vec{E}_{\text{in}}(r)$. Therefore, we can break the integral above over all

ⁱThe odd notation for volume dV_{ol} is to avoid confusion with electric potential without having to introduce more random Greek symbols. Some people will use $d\tau$ to represent volume to this end.

space into two separate integrals over each of these regions, representing the energy contained in the field outside the sphere, and that contained inside the sphere itself:

$$U = U_{\text{outside}} + U_{\text{inside}} = \frac{1}{8\pi k_e} \int_{\text{outside}} E_{\text{out}}^2 dV_{ol} + \frac{1}{8\pi k_e} \int_{\text{inside}} E_{\text{in}}^2 dV_{ol}$$

Obviously, the region outside the sphere is defined as $r > R$, and the region inside the sphere is $r \leq R$. Just like in the last example, we can perform the volume integral over spherical shells of radius r and width dr , letting the radius run from $0 \rightarrow R$ inside the sphere, and $R \rightarrow \infty$ outside. Our volume elements are thus $dV_{ol} = 4\pi r^2 dr$, just as in the previous method.

What about the fields? Outside the sphere, the electric field is just that of a point charge (thanks to Gauss' law). Noting again that the total charge on the sphere is $Q_{\text{tot}} = \frac{4}{3}\pi R^3 \rho$,

$$\vec{E}_{\text{out}}(r) = \frac{k_e Q_{\text{tot}}}{r^2} \hat{r} = \frac{4\pi k_e R^3 \rho}{3r^2} \hat{r} \quad (r > R)$$

Inside the sphere at a radius r , the electric field depends only on the amount of charge contained within a sphere of radius r , $q(r) = \frac{4}{3}\pi r^3 \rho$ as noted above, which we derived previously from Gauss' law:

$$\vec{E}_{\text{in}}(r) = \frac{k_e q(r)}{r^2} \hat{r} = \frac{4\pi k_e r \rho}{3} \hat{r} \quad (r \leq R)$$

Now all we have to do is perform the integrals ...

$$\begin{aligned} U &= U_{\text{outside}} + U_{\text{inside}} = \frac{1}{8\pi k_e} \int_{\text{outside}} E_{\text{out}}^2 dV_{ol} + \frac{1}{8\pi k_e} \int_{\text{inside}} E_{\text{in}}^2 dV_{ol} \\ &= \frac{1}{8\pi k_e} \int_R^{\infty} \left(\frac{4\pi k_e R^3 \rho}{3r^2} \right)^2 4\pi r^2 dr + \frac{1}{8\pi k_e} \int_0^R \left(\frac{4\pi k_e r \rho}{3} \right)^2 4\pi r^2 dr \\ &= \frac{8\pi^2 k_e R^6 \rho^2}{9} \int_R^{\infty} \frac{1}{r^2} dr + \frac{8\pi^2 k_e \rho^2}{9} \int_0^R r^4 dr \\ &= \frac{8\pi^2 k_e R^6 \rho^2}{9} \left[-\frac{1}{r} \right]_R^{\infty} + \frac{8\pi^2 k_e \rho^2}{9} \left[\frac{r^5}{5} \right]_0^R = \frac{8\pi^2 k_e \rho^2 R^5}{9} \left(\frac{1}{5} + 1 \right) \\ &= \frac{16\pi^2 k_e \rho^2}{15} R^5 \end{aligned}$$

Somehow, it is reassuring that these two different methods give us the same result ... or at least it should be!

Method 3: use $dU = \frac{1}{2} \rho V dV_{ol}$

As in the first method, we know that we can find the potential energy for any charge distribution by calculating the work required to bring in each bit of charge in the distribution in from an infinite distance away. For a collection of discrete point charges, we found:

$$U = \frac{1}{2} \sum_i \sum_{j \neq i} \frac{k_e q_i q_j}{r_{ij}} = \frac{1}{2} \sum_i \sum_{j \neq i} q_i V_j = \frac{1}{2} \sum_i \sum_{j \neq i} q_j V_i$$

Recall that the factor 1/2 is so that we don't count all pairs of charges twice. If we have a *continuous* charge distribution - like a sphere - our sum becomes an integral:

$$U = \frac{1}{2} \int_{\text{everywhere}} \rho V dV_{ol}$$

We still need the factor 1/2 to avoid double counting. Like the last example, we break this integral over all space up into two separate ones: one over the volume inside the sphere, and one outside. Outside the sphere at a distance $r > R$, the charge density ρ is zero, so that integral is zero. All we have to do is integrate the potential inside the sphere times the constant charge density over the volume of the sphere! But wait ... what is the potential inside the sphere?

The electric *field* inside such a sphere at a radius r we have already calculated, and it depended only on the amount of charge contained in a sphere of radius r , $q(r)$. That is because electric field from the spherical shell of charge more distant than r canceled out. This is *not* true of the potential, because the potential is a *scalar*, not a vector like the electric field, and there is no directionality that will cause the potential to cancel from the charge more distant than r .

Thus, the potential inside the sphere will actually have two components at a distance r inside the sphere: a term which looks like the potential of a point charge, due to the charge at a distance less than r from the center, and one that looks like the potential due to the remaining shell of charge from r to R . How do we calculate this mess? We can use the definition of the potential and the known electric field inside the sphere.

$$\vec{\mathbf{E}}_{in}(r) = \frac{k_e Q_{tot} r}{R^3} \hat{\mathbf{r}}$$

$$V_B - V_A = - \int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

In this case, we want to find the potential difference between a point at which we already know what it is - such as on the surface of the sphere - and a point r inside the sphere. Then we can use the known potential on the surface to find the unknown potential at r . We can most simply follow a radial path, $d\vec{\mathbf{l}} = \hat{\mathbf{r}} dr$. Since the electric field is conservative, the path itself doesn't matter, just the endpoints - so we always choose a very convenient path.

$$V(r) - V(R) = - \int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = - \int_R^r \frac{k_e Q_{tot} r}{R^3} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dr$$

$$= - \int_R^r \frac{k_e Q_{tot} r}{R^3} dr = - \frac{k_e Q_{tot}}{R^3} \left[\frac{r^2}{2} \right]_R^r$$

$$= \frac{k_e Q_{tot}}{2R^3} (R^2 - r^2)$$

We know what the potential at the surface $V(R)$ is, since outside the sphere the situation is identical to that of a point charge. Thus,

$$V(r) = V(R) + \frac{k_e Q_{\text{tot}}}{2R^3} (R^2 - r^2) = \frac{k_e Q_{\text{tot}}}{R} + \frac{k_e Q_{\text{tot}}}{2R^3} (R^2 - r^2) = \frac{k_e Q_{\text{tot}}}{2R} \left(3 - \frac{r^2}{R^2} \right)$$

Now we are ready! Once again, we integrate over our spherical volume by taking spherical shells of radius r and width dr , giving a volume element $dV_{\text{ol}} = 4\pi r^2 dr$. Noting that ρ is zero outside the sphere,

$$\begin{aligned} U &= \frac{1}{2} \int_{\text{everywhere}} \rho V dV_{\text{ol}} = U = \frac{1}{2} \int_0^R \rho V(r) 4\pi r^2 dr \\ &= \frac{1}{2} \int_0^R \frac{k_e Q_{\text{tot}} \rho}{2R} \left(3 - \frac{r^2}{R^2} \right) 4\pi r^2 dr = \frac{\pi k_e Q_{\text{tot}}}{R} \int_0^R 3r^2 - \frac{r^4}{R^2} dr \\ &= \frac{\pi k_e Q_{\text{tot}}}{R} \left[r^3 - \frac{r^5}{5R^2} \right]_0^R = \frac{\pi k_e Q_{\text{tot}}}{R} \left[R^3 - \frac{R^3}{5} \right] = \frac{4\pi k_e Q_{\text{tot}} \rho}{5} R^2 \\ &= \frac{16\pi^2 k_e \rho^2}{15} R^5 \end{aligned}$$

In the last line, we used $Q_{\text{tot}} = \frac{4}{3}\pi R^3 \rho$. One problem, three methods, and the same answer every time. Just how it ought to be. Which method should you use? It is a matter of taste, and the particular problem at hand. I tried to present the methods in order of what I thought was increasing difficulty, your opinion may differ.

2. *Griffiths 2.38, Jackson 2.5* An insulated spherical shell of radius a has a charge Q spread uniformly over its surface. If the sphere is cut evenly into two hemispheres, what force is required to hold the two hemispheres together?

There are also three reasonably simple solutions to this one. Once again, we'll tackle them in order of increasing pain.

Method 1: using mechanics + boundary conditions on \vec{E}

First off, forget about the charge. If this were just a spherical pressure vessel, evacuated to some pressure P relative to the outside atmosphere (at pressure P_o), what force would be required to separate the two hemispheres?

The pressure P on the surface of the two hemispheres is constant at all points, and the force F on each element of surface area is directed along the radius of the hemispheres. The applied force along the axis must balance the effective area over which the force acts. The effective area is *not* the whole surface area of the sphere, but the projection of the actual surface area onto the plane dividing the two hemispheres. That is, a circle of radius a . The force required to separate the spheres is just the pressure difference between the inside and the outside of the vessel multiplied by the effective area:

$$F = (P_o - P) \pi a^2$$

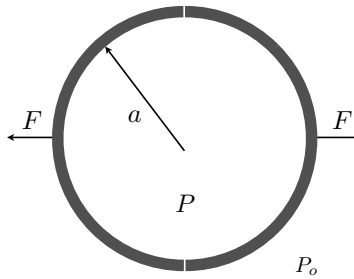


Figure 1: A spherical pressure vessel.

Hopefully, this is familiar from your mechanics class. With this result in hand, the problem is much easier. We derived in class that the force per unit area - which is just pressure - on a sheet of charge is related to the surface charge density, or the electric fields on either side of the sheet of charge:

$$\frac{F}{A} = P = 4\pi k_e \sigma = \frac{1}{8\pi k_e} (E_2^2 - E_1^2)$$

Here E_1 and E_2 are the electric fields on either side of the sheet of charge. In the present case, our "sheets" of charge are really hemispheres, so E_1 and E_2 represent the fields inside and outside the sphere, respectively. From Gauss' law, we know that inside the (unseparated) sphere, the electric field is zero, while outside the sphere the electric field is the same as that of a point charge Q . Just as in the case of a pressure vessel, the active area over which the electrostatic forces act is the projection of the hemispheres' area onto the plane separating them, or πa^2 . Thus,

$$F = PA = \frac{1}{8\pi k_e} (E_2^2 - E_1^2) (\pi a^2) = \frac{1}{8\pi k_e} \left[\left(\frac{k_e Q}{a^2} \right)^2 - 0 \right] (\pi a^2) = \frac{k_e Q^2}{8a^2}$$

Method 2: using $P = -\frac{dU}{dV}$

You may not remember - or perhaps you were never told - but there is a very nice relationship between pressure P , energy U , and volume V , which essentially equates pressure and energy density:

$$P = -\frac{dU}{dV}$$

You might have seen this in thermodynamics as $dW = PdV$. How do we use it? First, we have already derived the potential energy of a spherical shell of charge of radius r :

$$U(r) = \frac{k_e Q^2}{r}$$

What we want to do now is make the energy a function of *volume* rather than radius, noting that for a sphere of radius r ,

$$r = \sqrt[3]{\frac{3V}{4\pi}}$$

And so

$$U(V) = \frac{k_e Q^2}{2} \sqrt[3]{\frac{4\pi}{3V}} = \frac{k_e Q^2}{2} \sqrt[3]{\frac{4\pi}{3}} V^{-\frac{1}{3}}$$

We can now find the pressure by taking dU/dV :

$$P = -\frac{dU}{dV} = -\frac{k_e Q^2}{2} \sqrt[3]{\frac{4\pi}{3}} \left(\frac{-1}{3}\right) V^{-\frac{4}{3}} = \frac{k_e Q^2}{6} \sqrt[3]{\frac{4\pi}{3V^4}}$$

Finally, we know that the force corresponding to this pressure is $F = PA$, and from the previous approach we know the active area A is the projection of the hemispheres' area onto the plane separating them, or πr^2 .

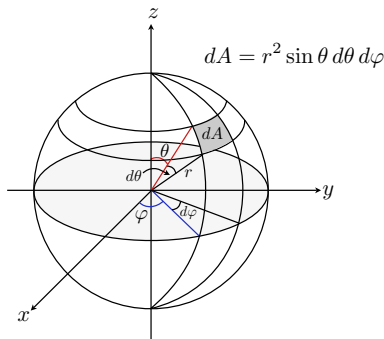
$$F = PA = \pi r^2 \frac{k_e Q^2}{6} \sqrt[3]{\frac{4\pi}{3V^4}} = \frac{\pi k_e Q^2 r^2}{6} \sqrt[3]{\frac{4\pi}{3\left(\frac{4}{3}\pi r^3\right)^4}} = \frac{\pi k_e Q^2 r^2}{6} \sqrt[3]{\frac{27}{64\pi^3 r^{12}}} = \frac{k_e Q^2}{8r^2}$$

Our particular sphere has a radius a , so $F = \frac{k_e Q^2}{8a^2}$ as we found with the previous method.

Method 3: Brute force integration.

Unless you have a bit of familiarity with calculus in three dimensions, you probably want to skip this one. In general we will not follow this sort of brute-force method, but I show it here as an example of how to solve problems like this almost from scratch, the way you will probably have to do it in PH331 or ECE340. So unless you are curious, go ahead and skip to the next problem.

Before we even try this, we should remind ourselves a little about spherical polar coordinates. This is just a three-dimensional version of polar coordinates - instead of (r, θ) defining a point within a plane, the coordinates (r, θ, φ) define a point in three dimensional space. Below you can see our conventions for choosing the r , θ , and φ "axes."



- r is the length of the vector
- θ is the angle with the positive z -axis; $0 \leq \theta \leq \pi$
- φ is the angle with the $x - z$ -plane; $0 \leq \varphi \leq 2\pi$
- differential surface area on a sphere is $dA = r^2 \sin \theta d\theta d\varphi$, with $\hat{n} = \hat{r}$
- differential volume is $dV = r^2 \sin \theta dr d\theta d\varphi$

Within what we would usually call the $x - y$ plane, we can locate any point with coordinates (r, φ) , where r is radial distance from the origin as usual, and φ is the angle the radius makes with respect to the x axis. We add to this is an angle θ to specify the inclination above the $x - y$ plane toward the z axis. In this system θ takes on values from 0 to π , and φ varies from 0 to 2π - to generate a sphere, we need a full rotation in one plane, but only half in the second.

So, for example, if we want to calculate the surface area of a sphere, we can first find the area of

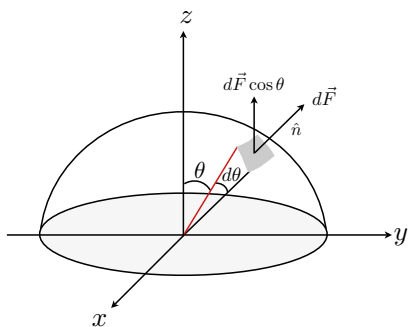
some tiny patch of the surface dA , defined by r , $d\theta$, and $d\varphi$, as shown above. If you remember your geometry (e.g., the formula for arclength), you should be able to show that the area dA is given by $r^2 \sin \theta d\theta d\varphi$ - one side of the patch of area is $r \sin \theta d\varphi$, the adjacent side is $r d\theta$, and if dA is small enough, we can treat it as a rectangle. The surface area of the whole sphere is then found by integrating each angular coordinate over its whole range. We do not need to integrate the radius, since that is constant for a sphere:

$$A = \int dA = \int_0^\pi d\theta \int_0^{2\pi} d\varphi r^2 \sin \theta = \int_0^\pi d\theta 2\pi r^2 \sin \theta = 2\pi r^2 [-\cos \theta]_0^\pi = 4\pi r^2$$

A familiar result, no? Compare this to integrating thin spherical shells to find volume, as we did above.

Now, on to the real problem at hand. What we really want to figure out is the electric force on a tiny patch of charge dA . In class, we derived that the force per unit area on a tiny sheet with charge density σ and fields E_1 on one side and E_2 on the other is:

$$\frac{F}{A} = \sigma E_{\text{avg}} = \frac{1}{2} \sigma (E_1 + E_2)$$



For a spherical shell of charge, E_1 and E_2 represent the fields inside and outside the sphere, respectively. From Gauss' law, we know that inside the (unseparated) sphere, the electric field is zero, while outside the sphere the electric field is the same as that of a point charge Q . By symmetry, the electric field and thus the electric force) must point radially outward from the center of the sphere. This is illustrated above for one hemisphere; note that the unit normal to dA is just $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, and $\hat{\mathbf{r}}$ makes an angle θ with the z axis. Knowing the total charge Q on the sphere and its radius a , for a tiny patch of area dA , the force dF must then be:

$$d\vec{F} = dF \hat{\mathbf{n}} = \frac{1}{2} \sigma (E_1 + E_2) dA \hat{\mathbf{r}} = \frac{1}{2} \sigma E_2 dA \hat{\mathbf{r}} = \frac{k_e Q^2}{8\pi a^4} dA \hat{\mathbf{r}} = \frac{k_e Q^2}{8\pi a^4} a^2 \sin \theta d\theta d\varphi \hat{\mathbf{r}}$$

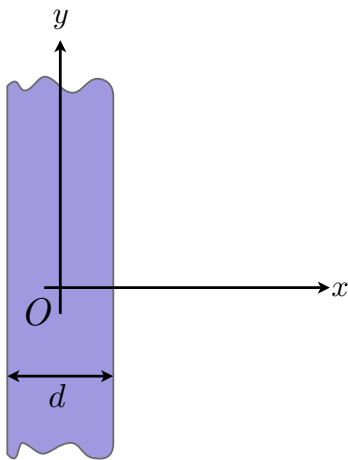
In the last step we used $\sigma = Q/4\pi a^2$. What is the force pushing this hemisphere away from the other one? Based on the symmetry of the system, we see that only the $\hat{\mathbf{z}}$ component of the force $d\vec{F}$ can result in a net displacement of the two hemispheres. In other words, only the z component of the force can do any work to separate the two hemispheres, and each bit dA is being pushed in the z direction by a force $d\vec{F} \cdot \hat{\mathbf{z}} = dF (\hat{\mathbf{r}} \cdot \hat{\mathbf{z}}) = dF \cos \theta$.

The total force acting on the entire hemisphere we can then find by integrating $dF \cos \theta$ over φ and

θ . In order to cover the hemisphere, θ must run from 0 to $\pi/2$, and φ from 0 to 2π .ⁱⁱ

$$\begin{aligned}
 F &= \int dF \cos \theta = \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi \left(\frac{k_e Q^2}{8\pi a^4} \right) (\cos \theta) a^2 \sin \theta \\
 &= \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi \frac{k_e Q^2}{8\pi a^2} \cos \theta \sin \theta = \frac{k_e Q^2}{8\pi a^2} \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi \sin \theta \cos \theta \\
 &= \frac{k_e Q^2}{8\pi a^2} \int_0^{\pi/2} d\theta 2\pi \sin \theta \cos \theta = \frac{k_e Q^2}{4a^2} \left[-\frac{\cos^2 \theta}{2} \right]_0^{\pi/2} = \frac{k_e Q^2}{8a^2}
 \end{aligned}$$

Sometimes it is satisfying to solve these things from very basic principles, but as you can see above, there are usually easier ways ...



3. *Serway 24.71* A slab of insulating material, infinite in two of its three dimensions, has a uniform positive charge density ρ , shown at left. **(a)** Show that the magnitude of the electric field a distance x from the center and inside the slab is $E = \rho x / \epsilon_0$. **(b)** Suppose an electron of charge $-e$ and mass m_e can move freely within the slab. It is released from rest at a distance x from the center. Show that the electron exhibits simple harmonic motion with a frequency

$$f = \frac{1}{2\pi} \sqrt{\frac{\rho e}{m_e \epsilon_0}}$$

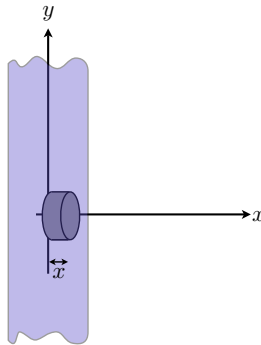
Consider a cylindrical shaped Gaussian surface perpendicular to the yz plane, as shown below. One circular face is in the yz plane, and the other is a distance x in the positive \hat{x} direction.

We know that for a thin slab of charge, the electric field must be perpendicular to the surface of the slab, or along the \hat{x} direction in this case. Further, by symmetry the field must be zero exactly at the center of the slab, where we have placed the left-most end cap. If we apply Gauss' law to this surface, the net flux is non-zero only through the rightmost end cap of the cylinder. Thus, if we call the area of the cylinder's end cap $A_{\text{end cap}}$,

$$\oint \vec{E} \cdot \hat{n} dA = EA_{\text{end cap}}$$

The amount of charge contained in the Gaussian cylinder is just the charge density of the slab times the volume of the cylinder, $x A_{\text{end cap}}$, so

ⁱⁱIf we let θ run over the full range of 0 to π , we would integrate over the whole sphere.



$$EA_{\text{end cap}} = \frac{\rho A_{\text{end cap}} x}{\epsilon_0} \quad \implies \quad E = \frac{\rho x}{\epsilon_0}$$

Fair enough. What is the force on an electron placed at a distance x ? We know it is the electron's charge $-e$ times the electric field at x , and if this is the only force acting, it must equal the electron's mass m_e times the resulting acceleration.

$$\vec{\mathbf{F}} = -e\vec{\mathbf{E}} = \frac{-e\rho x}{\epsilon_0} \hat{\mathbf{x}} = m_e \vec{\mathbf{a}}$$

Thus, the acceleration must be in the $\hat{\mathbf{x}}$ direction, and it is proportional to the electron's position x . How do we show that simple harmonic motion results? We just did! Recall from mechanics that simple harmonic motion occurs whenever we can show that a system's acceleration are related by some constant $-\omega^2$, viz., $a = -\omega^2 x$.ⁱⁱⁱ If this is true, then ω is the angular frequency of oscillation, and $\omega = 2\pi f$. Rewriting the equation above, dropping the now redundant vector notation,

$$a = \frac{d^2x}{dt^2} = -\frac{e\rho}{m_e\epsilon_0}x$$

$$\implies \quad \omega = \sqrt{\frac{e\rho}{m_e\epsilon_0}} \quad \text{and} \quad f = \frac{1}{2\pi} \sqrt{\frac{e\rho}{m_e\epsilon_0}}$$

4. *Serway 23.24* Consider an infinite number of identical charges (each of charge q) placed along the x axis at distances $a, 2a, 3a, 4a, \dots$ from the origin. **(a)** What is the electric field at the origin due to this distribution? **(b)** What is the electric potential at the origin?

We have identical charges q placed at distances which are integer multiples of a from the origin. The total electric field is the sum of each individual electric field from each individual charge. Since all charges are along the x axis, their fields all point in the $-\hat{\mathbf{x}}$ direction, and we can simply add the magnitudes.

ⁱⁱⁱSomewhat more precisely, we want to show that the differential equation $d^2x/dt^2 = -\omega^2x$ is obeyed, which has the simplest physical solution $x(t) = A \cos(\omega t + \delta)$.

If we number these charges sequentially, starting from the closest one and continuing until the n^{th} , the magnitude of the electric fields from the first few charges can be calculated easily, and we begin to notice a pattern:

$$\begin{aligned} E_1 &= -\frac{k_e q}{a^2} \\ E_2 &= -\frac{k_e q}{(2a)^2} = -\frac{k_e q}{4a^2} \\ E_3 &= -\frac{k_e q}{(3a)^2} = -\frac{k_e q}{9a^2} \\ &\vdots \\ E_n &= -\frac{k_e q}{(na)^2} = -\frac{k_e q}{n^2 a^2} \end{aligned}$$

The minus sign is to indicate that the field is in the $-\hat{x}$ direction. Having noticed a pattern, we can more easily represent the total electric field as an infinite sum, noting that the n^{th} charge from the origin is at a distance na , and all charges are identical:

$$E_{\text{tot}} = \sum_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} \frac{k_e q_n}{r_n^2} = \sum_{n=1}^{\infty} \frac{k_e q}{(na)^2} = \frac{k_e q}{a^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

This infinite sum is convergent, and has the remarkably simple value of $\pi^2/6$.^{iv} The total electric field is then

$$E_{\text{tot}} = \frac{k_e q}{a^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{k_e q}{a^2} \left(\frac{\pi^2}{6} \right) = \frac{\pi^2 k_e q}{6a^2}$$

What about the electric potential? The potential also obeys superposition, so our sum now looks like this:

$$V_{\text{tot}} = \sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{k_e q_n}{r_n} = \sum_{n=1}^{\infty} \frac{k_e q}{na} = \frac{k_e q}{a} \sum_{n=1}^{\infty} \frac{1}{n}$$

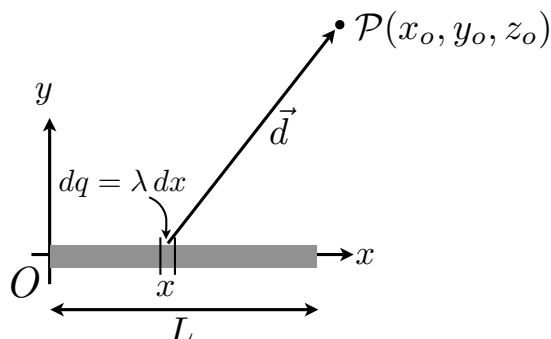
This sum is known as the harmonic series. Sadly, this innocent-looking series diverges. The potential is infinite, even though the field is finite, which means one important thing: such an arrangement of charges could never be assembled, since it would take an infinite amount of energy!

5. *Serway 24.68* Find the electric potential at an arbitrary point $P(x, y, z)$ in the vicinity of a charged rod of length L with linear charge density λ .

First, we note that Gauss' law cannot be used easily, since the rod is of finite length - symmetry will not really help us, since the field will vary strongly near the end points. We could find the electric field at point P (similar to what you did in HW 1) and then find its gradient, but that seems like a lot

^{iv}There are some very clever proofs of this, my favorite perhaps being the one using Fourier series and Parseval's identity. The first proof was due to Euler: http://en.wikipedia.org/wiki/Basel_problem

of work. Best is just to brute force this one, it ends up working out fairly easily. Start with the figure below:



We place the rod along the x axis, with one end at the origin. We will only draw the $x - y$ plane here, but will consider distances in all three dimensions. If we consider some tiny slice of the rod dx , it must contain a charge $dq = \lambda dx$ if the charge density is uniform. What we want to do is find the potential at point P due to this dq , and then integrate over all possible dq 's.

Let our arbitrary point P be at the coordinates (x_o, y_o, z_o) . For an arbitrary element dq at $(x, 0, 0)$, the position vector \vec{d} and distance d from dq to point P is

$$\vec{d} = (x_o - x) \hat{x} + y_o \hat{y} + z_o \hat{z} \quad \implies \quad d = \sqrt{(x_o - x)^2 + y_o^2 + z_o^2}$$

Since our infinitesimal charge dq can be considered a point charge, the potential dV at point P due dq is just

$$dV(x_o, y_o, z_o) = \frac{k_e dq}{d} = \frac{k_e \lambda dx}{\sqrt{(x_o - x)^2 + y_o^2 + z_o^2}}$$

The total potential at point P is then just integrating all of these dq , which means integrating dx over the length of the rod from 0 to L :

$$\begin{aligned} V(x_o, y_o, z_o) &= \int dV(x_o, y_o, z_o) = \int_0^L \frac{k_e \lambda}{\sqrt{(x_o - x)^2 + y_o^2 + z_o^2}} dx \\ &= k_e \lambda \left[\ln \left| x_o - x + \sqrt{(x_o - x)^2 + y_o^2 + z_o^2} \right| \right]_0^L \\ &= k_e \lambda \left[\ln \left| x_o - L + \sqrt{(x_o - L)^2 + y_o^2 + z_o^2} \right| - \ln \left| x_o + \sqrt{x_o^2 + y_o^2 + z_o^2} \right| \right] \\ &= k_e \lambda \ln \left| \frac{|x_o - L| + \sqrt{(x_o - L)^2 + y_o^2 + z_o^2}}{x_o + \sqrt{x_o^2 + y_o^2 + z_o^2}} \right| \end{aligned}$$

In the limiting case that we are at $P = (0, a, 0)$, we recover the result of one of the textbook's example problems (Ch. 25):

$$V(0, a, 0) = k_e \lambda \ln \left(\frac{L + \sqrt{L^2 + a^2}}{a} \right) = k_e \frac{Q}{L} \ln \left(\frac{L + \sqrt{L^2 + a^2}}{a} \right)$$

Optional: *putting the result in a coordinate-free form.* Now define $|\vec{r}_o| = r_o = \sqrt{x_o^2 + y_o^2 + z_o^2}$, which is just the distance from the origin to P . Next, define a vector $\vec{L} = L \hat{x}$ which points from one end of the rod to the other, as well as its corresponding unit vector $\hat{L} = \vec{L}/|\vec{L}|$. With those definitions, we notice that the vector from the right end of the rod to P can be written as $\vec{r}_o - \vec{L}$.

Now we can re-write the distances as vector differences. For example, $\sqrt{(x_o - L)^2 + y_o^2 + z_o^2} = |\vec{r}_o - \vec{L}|$, and $x_o - L = (\vec{r}_o - \vec{L}) \cdot \hat{L}$, and finally $x_o + r_o = \vec{r}_o \cdot \hat{L} + |\vec{r}_o|^2$. Why bother? Because now we can write our answer in a coordinate-free form, which does not depend on our choice of origin, coordinate system, or anything:

$$V = k \lambda \ln \left| \frac{(\vec{r}_o - \vec{L}) \cdot \hat{L} + |\vec{r}_o - \vec{L}|}{\vec{r}_o \cdot \hat{L} + |\vec{r}_o|^2} \right|$$

The potential depends only on the *relative* positions of the rod and P , and the angle between \vec{r} and \vec{L} , as we expect. (The angle basically tells us whether P is more along the rod's axis, or more along a perpendicular to its axis.)

6. *Purcell 2.5* A sphere the size of a basketball is charged to a potential of -1000 V. About how many extra electrons are on it, per cm^2 of surface?

If the charge is spread evenly over the surface of a spherical object, like a basketball, then Gauss' law says we may treat the charge distribution as a point charge. Thus, we may consider the uniformly charged basketball equivalent to a single point charge at the center of the basketball. A men's regulation basketball has a diameter of 9.39 in, and thus a radius of 0.12 m. We know then that the potential at the surface of the basketball is -1000 V, and this potential results from an effective point charge q at the center of the basketball, 0.12 m from the surface. Using the expression for the potential from a point charge:

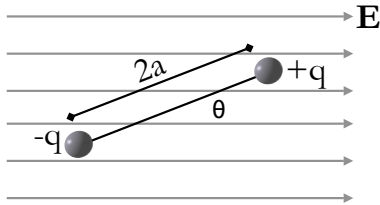
$$\begin{aligned} -1000 \text{ V} &= \frac{k_e q}{r} = \frac{[9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2] q}{0.12 \text{ m}} \\ \implies q &= -1.33 \times 10^{-8} \text{ C} \end{aligned}$$

This is the effective charge on the basketball. Given that one electron has -1.6×10^{-19} C of charge, this must correspond to:

$$(\text{number of electrons}) = \frac{-1.33 \times 10^{-8} \text{ C}}{-1.6 \times 10^{-19} \text{ C/electron}} = 8.33 \times 10^{10} \text{ electrons}$$

If these electrons are spread out evenly over the surface, the electron density can be calculated from the surface area of the basketball (remembering that the surface area of a sphere is $4\pi r^2$, and we were asked to use cm^2):

$$(\text{density of electrons}) = \frac{8.33 \times 10^{10} \text{ electrons}}{4\pi (12 \text{ cm})^2} \approx 4.6 \times 10^7 \text{ electrons/cm}^2$$



7. *Serway 23.73* An electric dipole in a uniform electric field E is displaced slightly from its equilibrium position, as shown at left, where θ is small. The separation of the charges is $2a$, and the moment of inertia of the dipole is I . Assuming the dipole is released from this position, show that its angular orientation exhibits simple harmonic motion with a frequency

$$f = \frac{1}{2\pi} \sqrt{\frac{2qaE}{I}}$$

Define the positive x axis to be in the direction of the electric field, and the positive y axis perpendicular to it in the upward direction. This means the z axis points out of the page for a right-handed coordinate system.

On the $+q$ charge, there will be force $F_+ = qE$ along \hat{x} , and on the $-q$ charge, a force $F_- = -qE$ (along $-\hat{y}$). Both of these forces will try to cause the dipole to rotate and orient itself along the electric field; that is, both will result in a clockwise torque about the center of the dipole. The sum of these torques must equal the dipole's moment of inertia I times the resulting angular acceleration α . Let the position of the $+q$ charge be defined by a vector \vec{r}_+ whose origin is at the dipole center, and similarly \vec{r}_- will give the position of the $-q$ charge. We also define a unit vector \hat{r} pointing from the $-q$ to the $+q$ charge. Finally, remember that a clockwise rotation defines a *negative* torque - this is the version of the right-hand rule for torques.^v

$$\begin{aligned} \sum \vec{\tau} &= (\vec{r}_+ \times \vec{F}_+) + (\vec{r}_- \times \vec{F}_-) = qEa \hat{r} \times \hat{x} + (-qE)(-\hat{r} \times \hat{x}) \\ &= qEa(-\hat{z} \sin \theta) + qEa(-\hat{z} \sin \theta) = -2qaE \sin \theta \hat{z} = I\vec{\alpha} \\ \sum |\vec{\tau}| &= -2qaEa \sin \theta = I|\vec{\alpha}| \end{aligned}$$

We could have avoided the vector baggage right off the bat, if we just chose the resultant torque to be negative based on the right-hand rule. In order to show simple harmonic motion, we need to show in this case that $\alpha = -\omega^2\theta$. Recalling the definition of α , we have:

$$I\alpha = I \frac{d^2\theta}{dt^2} = -2qaEa \sin \theta$$

If the angle θ is small, we can approximate $\sin \theta$ by the first term in its Taylor expansion (a "first-order" approximation):

^vWe have to pick the sign because $\vec{\tau}$, resulting from a cross-product, is a pseudovector.

$$\begin{aligned} \text{Taylor expansion} \quad \sin \theta &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \theta^{2n+1} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \\ \text{small } \theta : \quad \sin \theta &\approx \theta \end{aligned}$$

Using this approximation,

$$I \frac{d^2\theta}{dt^2} \approx -2qEa\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} \approx -\frac{2qEa}{I} \theta$$

This is our beloved differential equation for simple harmonic motion, viz., $a = -\omega^2 x$, and thus for small θ ,

$$\omega = \sqrt{\frac{2qaE}{I}} \quad \text{or} \quad f = \frac{1}{2\pi} \sqrt{\frac{2qaE}{I}}$$