

Problem Set 8: Solutions

1. (*Purcell 7.22*) A thin ring of radius a carries a static charge q . This ring is in a magnetic field of strength B_o , parallel to the ring's axis, and is supported so that it is free to rotate about that axis. If the field is switched off, how much angular momentum will be added to the ring? If the ring has mass m , show that it will acquire an angular velocity $\omega = qB_o/2m$.

It is not so clear, right off the bat, why the ring should start to rotate in the first place. Where can the angular momentum come from? The key is to recognize that time-varying magnetic field will give rise to a *circulating* electric field, which will cause a torque on the ring. One step at a time though!

At the instant the field is switched off, the magnetic flux through the loop changes from $\Phi_B = B_o\pi a^2$ to zero. The fact that the flux changes in time leads to an induced voltage around the ring, by Faraday's law:

$$\frac{\partial\Phi_B}{\partial t} = -\Delta V = \oint_{\text{ring}} \vec{E} \cdot d\vec{l}$$

If there is an induced voltage around the ring, then there must be an electric field as well, since the integral of $\vec{E} \cdot d\vec{l}$ around the loop is nonzero. What is the direction of the electric field? If the original B field was constant along the ring's axis, then the electric field resulting from its disappearance must be circulating. If the ring were conducting - and we did not say that it is - the change in magnetic flux would create induced currents circulating in such a way to stop the decrease in flux. A circulating current in the ring could only be caused by an electric field varying tangentially along the ring, i.e., a circulating electric field. The situation is the same whether the ring is conducting or not, in that the electric field will circulate around the axis of the ring.

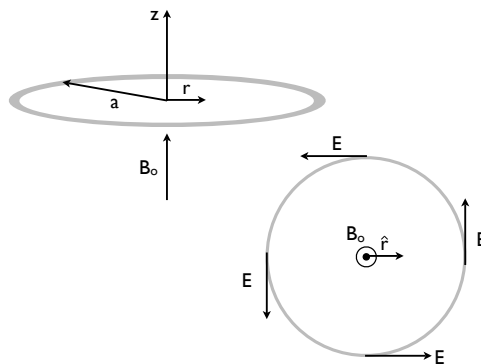


Figure 1: Problem 1: If one field is 'straight,' the other circulates.

The change in magnetic flux produces an electric field tangent to the ring everywhere. This tangential electric field is what will give rise to a torque. Each tiny element of the ring dl will feel an electric field

E , and thus an electric force F_e in the tangential direction. Over the whole ring, adding up many tiny segments, there will be no net force, but there will be a net torque about the z axis.

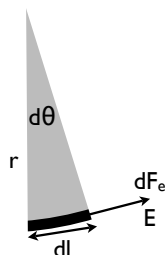


Figure 2: Problem 1: a tiny segment of the ring.

The ring of radius a has a total charge q spread out uniformly. Let us define a linear charge density $\lambda = q/2\pi a$. Any tiny element dl of the ring then has a charge $dq = \lambda dl$. The electric field E will give rise to an electric force on each segment, also in the tangential direction:

$$d\vec{F}_e = dq \vec{E} = \lambda dl E \hat{\theta}$$

On any little segment, we can now readily find the torque, since the electric force acts at a right angle to the radial direction. Remember that the ring is supported at its center point, which means the electric force acts over a distance a from the center of the circle to its rim.

$$d\vec{\tau} = \vec{a} \times d\vec{F}_e = a \hat{r} dl E \hat{\theta} = \lambda a dl E (\hat{r} \times \hat{\theta}) = \lambda a E dl \hat{z}$$

The torque is in the z direction, consistent with the right-hand rule. Once we know the torque on a small element dl , we can easily integrate to find the torque on the whole ring:

$$\vec{\tau} = \oint_{\text{ring}} d\vec{\tau} = \oint \lambda a E dl \hat{z}$$

Now we should notice two things: first, that the radius r is constant, and can be taken out of the integral. Second, since \vec{E} and $d\vec{l}$ are parallel everywhere, the integral of $E dl$ around the loop is nothing more than the potential difference around the ring ΔV !

$$\vec{\tau} = \lambda a \oint E dl \hat{z} = \lambda a \Delta V \hat{z} = \left(\frac{q}{2\pi a} \right) a \Delta V \hat{z} = \left(\frac{q \Delta V}{2\pi} \right) \hat{z}$$

The torque on the ring is simply proportional to the induced voltage, which is in turn proportional to the change in magnetic flux. We will drop the vector notation now, since we have the directions of everything straight at this point.

$$\tau = \left(\frac{q \Delta V}{2\pi} \right) = - \left(\frac{q}{2\pi} \right) \frac{d\Phi_B}{dt}$$

Of course, what is torque but the time rate of change of angular momentum ...

$$\frac{dL}{dt} = - \left(\frac{q}{2\pi} \right) \frac{d\Phi_B}{dt}$$

Now integrate both sides. Let the limits of integration be the state i , just before the field B_o is switched off, and the state f just after the field has been switched off.

$$\int_i^f dL = - \frac{q}{2\pi} \int_i^f d\Phi_B \quad \implies \quad L_f - L_i = \frac{q}{2\pi} (\Phi_{B,i} - \Phi_{B,f})$$

Just before the field is switched off, we have no net angular momentum, so $L_i = 0$, and just after the field is switched off, we have $\Phi_{B,f} = 0$. Since we know the initial magnetic flux as well,

$$L_f = \frac{q}{2\pi} (B_o \pi a^2) = \frac{1}{2} q a^2 B_o$$

So there is an angular momentum after the field is switched off, and its magnitude scales with the size of the initial field. In another way, we can say that the electric and magnetic fields also have angular momentum, just as they have an energy density. In fact, the fields carry energy, momentum, and angular momentum, a fact which is most easily seen from their effect on simple charge distributions like this one.

What about the rate of rotation? The net angular momentum for an object must equate to its moment of inertia I and rate of angular rotation ω : $\sum L = I\omega$. For a thin circular ring of mass m and radius a , we know that $I = ma^2$.ⁱ This is the final thing we need:

$$L_f = I\omega = \omega ma^2 = \frac{1}{2} q a^2 B_o \quad \implies \quad \omega = \frac{q B_o}{2m}$$

In the end, ω depends only on the initial field strength, and the charge/mass ratio of the ring – not at all on the size of the ring.

2. (*Purcell 7.23*) There is evidence that a magnetic field exists in most of the interstellar space with a strength between 10^{-9} and 10^{-10} T. Adopting 3×10^{-10} T as a typical value, find the total energy stored in the magnetic field of the galaxy. Assume the galaxy is a disk roughly 10^{21} m in diameter and 10^{19} m thick. Assuming stars radiate about 10^{37} W, how many years of starlight is the magnetic energy worth?

Don't over-think this one. It is easier than it sounds. The energy per unit volume contained in the magnetic field is $B^2/2\mu_o$. The total energy contained in the magnetic field over some volume is:

$$U = \frac{1}{2\mu_o} \int B^2 dV$$

If we assume that the interstellar magnetic field has a constant value B_o , the volume integral is trivial, and the total energy contained in the field in a volume V is just $B_o^2 V / 2\mu_o$. All we need to do is find the volume of the galaxy! If we model the galaxy as a disk of *diameter* 10^{21} m and thickness 10^{19} m, then

ⁱOf course by "know" we mean "you can look these things up."

$$V = \pi \left(\frac{1}{2} \times 10^{21} \text{ m} \right) (10^{19} \text{ m}) \approx 8 \times 10^{60} \text{ m}^3$$

The total energy stored in the magnetic field is then

$$U = \frac{\mu_o B_o^2}{2} V \approx 3 \times 10^{47} \text{ J}$$

If all stars radiate about 10^{37} W , since $\mathcal{P} = dU/dt$, the time Δt it takes to radiate away all the energy is about

$$\Delta t = \frac{U}{\mathcal{P}} \approx \frac{\times 10^{47} \text{ J}}{10^{37} \text{ J/s}} = 3 \times 10^{10} \text{ s} \sim 900 \text{ yrs}$$

3. *Purcell 9.10* Find the magnetic field at a point P midway between the plates of capacitor a distance r from the axis of symmetry. A current I is flowing through the capacitor.

We assume that the magnetic field circles the capacitor axis, consistent with the magnetic field flowing through the wires leading to the capacitor. If the capacitor spacing s is small, the electric field will be fairly uniform. Let the plates be squares of side b to be concrete.

$$E = \frac{\Delta V}{s} = \frac{Q}{sC} = \frac{Qs}{s\epsilon_o b^2} = \frac{Q}{\epsilon_o b^2}$$

The electric flux through a circle of radius r , through the point of interest, is

$$\Phi_E = \pi r^2 E = \frac{\pi r^2 Q}{b^2},$$

The time rate of change of the electric flux gives us the magnetic field:

$$\int \vec{B} \cdot d\vec{l} = 2\pi r B = \frac{1}{c^2} \frac{d\Phi_E}{dt} = \frac{1}{c^2} \frac{\pi r^2}{\epsilon_o b^2} \frac{dQ}{dt},$$

Rearranging, and noting $\frac{1}{c^2} = \epsilon_o \mu_o$,

$$B = \frac{r}{2c^2 \epsilon_o b^2} \frac{dQ}{dt} = \frac{\mu_o r}{2b^2} \frac{dQ}{dt}$$

Finally, conservation of charge dictates that the time rate of change of charge going to or from the capacitor must be the same as the current flowing: $I = dQ/dt$:

$$B = \frac{\mu_o I r}{2b^2}$$

At the edge of the capacitor ($r = b$), this is the same as the magnetic field around a long wire.

4. Serway 32.69 At $t = 0$, the open switch in the figure below is closed. By using Kirchhoff's rules for the instantaneous currents and voltages in this two-loop circuit, show that the current in the inductor at time $t > 0$ is

$$I(t) = \frac{\Delta V}{R_1} \left[1 - e^{-(R'/L)t} \right]$$

where $R' = R_1 R_2 / (R_1 + R_2)$

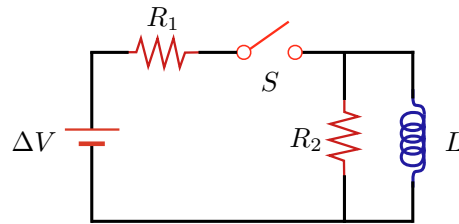


Figure 3: Problem 4: an RL circuit.

We should note that this circuit looks a lot like a "realistic" version of our ideal RL circuit - the battery has some internal resistance R_1 , and the inductor has some finite resistance R_2 . This may not be such a silly problem.

Let a current I_2 flow through resistor R_2 , a current I through the inductor. Conservation of charge means that a current $I_2 + I$ must flow through resistor R_1 (and through the battery ΔV). First, apply conservation of energy to the loop containing R_1 and R_2 :

$$\Delta V - (I + I_2) R_1 - I_2 R_2 = 0$$

Next, apply conservation of energy to the loop containing R_2 and L :

$$\Delta V - (I + I_2) R_1 - L \frac{dI}{dt} = 0$$

We can solve the first equation for I_2 , the current in resistor 2, and plug that into the second equation. Step one:

$$\begin{aligned} \Delta V - (I + I_2) R_1 - I_2 R_2 &= 0 \\ I_2 (R_1 + R_2) &= \Delta V - I R_1 \\ I_2 &= \frac{\Delta V - I R_1}{R_1 + R_2} \end{aligned}$$

Step two, substitute and rearrange:

$$\begin{aligned} \Delta V - (I + I_2) R_1 - L \frac{dI}{dt} &= 0 \\ \Delta V - IR_1 - I_2 R_1 - L \frac{dI}{dt} &= 0 \\ \Delta V - IR_1 - \left(\frac{\Delta V - IR_1}{R_1 + R_2} \right) R_1 - L \frac{dI}{dt} &= 0 \\ \Delta V - \Delta V \left(\frac{R_1}{R_1 + R_2} \right) - IR_1 + \frac{IR_1^2}{R_1 + R_2} - L \frac{dI}{dt} &= 0 \\ \Delta V \left(1 - \frac{R_1}{R_1 + R_2} \right) - IR_1 \left(1 - \frac{R_1}{R_1 + R_2} \right) - L \frac{dI}{dt} &= 0 \\ \Delta V \left(\frac{R_2}{R_1 + R_2} \right) - IR_1 \left(\frac{R_2}{R_1 + R_2} \right) - L \frac{dI}{dt} &= 0 \end{aligned}$$

How do we proceed from here? We want to make this equation look like one we've already solved, namely, the equation for a simple LR circuit. First, define a quantity $\Delta V'$, which is just the first term in our equation above:

$$\Delta V' = \Delta V \frac{R_2}{R_1 + R_2}$$

This term is just the potential difference we would have across R_2 if the inductor were not present. Next, define a quantity R' , which is most of the second term in our main equation:

$$R' = \frac{R_1 R_2}{R_1 + R_2}$$

This is the same as a parallel combination of R_1 and R_2 . Now rewrite our main equation:

$$\begin{aligned} \Delta V \left(\frac{R_2}{R_1 + R_2} \right) - IR_1 \left(\frac{R_2}{R_1 + R_2} \right) - L \frac{dI}{dt} &= 0 \\ \Delta V' - IR' - L \frac{dI}{dt} &= 0 \end{aligned}$$

Aha! This is the same differential equation for an LR circuit we already solved. We never solve equations more than once ... the same equations have the same solutions, so we can just write down the result:

$$I(t) = \frac{\Delta V'}{R'} \left(1 - e^{-R't/L} \right)$$

Now notice that $\Delta V'/R' = \Delta V/R_1$, something you can easily verify, and we have our final answer:

$$I(t) = \frac{\Delta V}{R_1} \left(1 - e^{-R't/L} \right)$$

5. *Purcell* 10.7 A cell membrane typically has a capacitance around $1 \mu\text{F}/\text{cm}^2$. It is believed the membrane

consists of material having a dielectric constant of $\kappa \sim 3$. Find the thickness this implies. Other electrical measurements have indicated that the resistance of 1 cm^2 of cell membrane is around 1000Ω . Show that the time constant of such a leaky capacitor is independent of the area of the capacitor. How large is it in this case? What is the resistivity?

Our first key assumption is that we can model our cell membrane as a parallel plate capacitor. This seems outlandishly stupid on first sight, but in fact it is a reasonable starting point for modeling the phospholipid bilayer in a cell membrane. The polar head groups on the lipid molecules in the lipid bilayer do form an extended structure that resembles a capacitor, if we think about the charge transfer involved. Anyway: if you are not up on your microbiology (and it has been a while for me), just take this starting point for granted for now.

This problem is a lot easier than it looks if we just write down the expression for capacitance per unit area. That is what we are given, and sometimes when you aren't quite sure what to do, just write down what you know. Let the interior of the cell membrane have a thickness d and a relative dielectric constant κ . Then,

$$\frac{C}{A} = \frac{\kappa\epsilon_o}{d} = 1 \mu\text{F}/\text{cm}^2$$

As it turns out, capacitance per unit area depends only on the spacing between charges, in this case the thickness of the cell membrane, so we can find d easily:

$$d = \frac{\kappa\epsilon_o}{C/A} \approx 2.7 \times 10^{-9} \text{ m}$$

Be careful with the units! This is about 3 nm, which seems about the right order of magnitude for the length of a small lipid molecule. What about our leaky capacitor? All this means is that the capacitor lets some dc current through it, as if it had a resistor in *parallel* that let some current "leak" around it. In other words, a "leaky" capacitor is just a parallel RC circuit.

The resistive part of the circuit we can model as a cylinder of area A and thickness d , which has a resistance of $R = \rho d/A$. Notice that the *resistance area product* is just $RA = \rho d$. Now, the time constant of a parallel RC circuit is just $\tau = RC$. This is the same thing as multiplying the capacitance per unit area and the resistance area product: $\tau = RC = (RA)(C/A)$. Basically: we can calculate the time constant of our leaky capacitor without knowing its area.

$$\tau = RC = (RA) \left(\frac{C}{A} \right) = \left(\frac{\rho d}{A} \right) \left(\frac{\kappa\epsilon_o A}{d} \right) = \rho\kappa\epsilon_o$$

Independent of area, as desired, and also independent of thickness! Given the value of the resistance R for a specific area A , we can find the area product RA and calculate the resistivity:

$$\rho = \frac{RA}{d} \approx 4 \times 10^7 \Omega \cdot \text{m}$$

Given a resistivity, we can get a number for the time constant, $\tau \approx 10^{-3} \text{ s}$.

6. Serway 32.45 Two inductors having self-inductance L_1 and L_2 are connected in parallel to a time-

varying source of current $I(t)$. The mutual inductance between the two inductors is M_{12} . Determine the equivalent self-inductance L_{eq} for the system.

Since our two inductors are in parallel, they must have the same voltage, but they will in general have different currents. We can write the voltage on either in a very general way as the sum of two terms: the self inductance, which for a given inductor gives a contribution proportional to the rate of change of its own current; and the mutual inductance, proportional to the current through the *other* inductor. Call the source voltage ΔV , the inductance of the two inductors L_1 and L_2 , the current in each I_1 and I_2 , and their mutual inductance M . The general expression for the voltage on either inductor is:

$$\Delta V = -L_1 \frac{dI_1}{dt} - M \frac{dI_2}{dt} \quad (1)$$

$$\Delta V = -L_2 \frac{dI_2}{dt} - M \frac{dI_1}{dt} \quad (2)$$

Since the two inductors are in parallel, we know that their individual currents must total the current I leaving the source: $I = I_1 + I_2$. What we would like to do is come up with an equation relating the time derivative of the *total* current, dI/dt , to the source voltage, such that we can find an effective inductance: $\Delta V = -L_{eq}dI/dt$. To start with, we can rearrange equations 1 and 2:

$$-\frac{dI_1}{dt} = \frac{\Delta V}{L_1} + \frac{M}{L_1} \frac{dI_2}{dt} \quad (3)$$

$$-\frac{dI_2}{dt} = \frac{\Delta V}{L_2} + \frac{M}{L_2} \frac{dI_1}{dt} \quad (4)$$

Note the nice symmetry of our equations - we can basically just swap 1 and 2 to get from one to the other. We can use this later to save a bit of work. First, plug equation 3 into equation 2 and simplify:

$$\begin{aligned} \Delta V &= -L_2 \frac{dI_2}{dt} - M \frac{dI_1}{dt} = -L_2 \frac{dI_2}{dt} + M \left[\frac{\Delta V}{L_1} + \frac{M}{L_1} \frac{dI_2}{dt} \right] \\ \Delta V - \Delta V \left[\frac{M}{L_1} \right] &= \left[\frac{M^2}{L_1} - L_2 \right] \frac{dI_2}{dt} \\ \Delta V \left[1 - \frac{M}{L_1} \right] &= \left[\frac{M^2}{L_1} - L_2 \right] \frac{dI_2}{dt} \\ \Delta V [L_1 - M] &= [M^2 - L_1 L_2] \frac{dI_2}{dt} \\ \implies \frac{dI_2}{dt} &= \Delta V \left[\frac{L_1 - M}{M^2 - L_1 L_2} \right] \end{aligned} \quad (5)$$

Now we have an expression for dI_2/dt which does not involve I_1 . We can follow exactly the same procedure, substituting equation 4 into equation 1. Since the equations are nicely symmetric, you should be able to write down the answer by inspection after completing the work above:

$$\frac{dI_1}{dt} = \Delta V \left[\frac{L_2 - M}{M^2 - L_1 L_2} \right] \quad (6)$$

For the final step, add equations 5 and 6 together, and remember $I = I_1 + I_2$:

$$\begin{aligned}
\frac{dI_1}{dt} + \frac{dI_2}{dt} &= \Delta V \left[\frac{L_1 - M}{M^2 - L_1 L_2} \right] + \Delta V \left[\frac{L_2 - M}{M^2 - L_1 L_2} \right] \\
\frac{dI}{dt} &= \Delta V \left[\frac{L_1 + L_2 - 2M}{M^2 - L_1 L_2} \right] \\
\Delta V &= - \left[\frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M} \right] \frac{dI}{dt} \equiv -L_{eq} \frac{dI}{dt} \quad \text{with} \quad L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}
\end{aligned}$$

It is interesting to notice that if the two inductors are well isolated, such that $M = 0$, the parallel inductors add just like parallel resistors. The same happens for series inductors - if they are decoupled ($M = 0$), they add just like resistors.