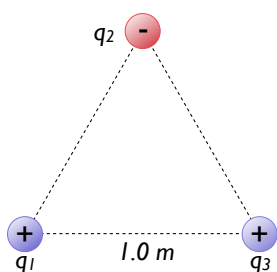


Problem Set 10: Exam Review

Instructions:

1. This homework set is not graded, but merely a review for the final exam
2. Some problems may show up verbatim on the final exam



1. Three charges are arranged in an equilateral triangle, as shown at left. All three charges have the same *magnitude* of charge, $|q_1| = |q_2| = |q_3| = 10^{-9}$ C (note that q_2 is negative though). What is the total **potential energy** of this system of charges? Take the zero of potential energy to be when all charges are infinitely far apart.

–9 nJ. The potential energy of a system of charges can be found by calculating the potential energy for every unique pair of charges and adding the results together. In this case, we have three unique pairings: charges 1 and 2, charges 2 and 3, and charges 1 and 3:

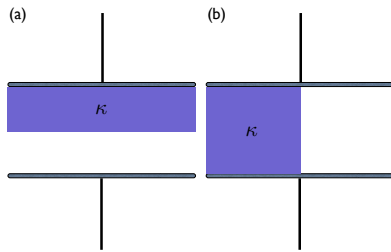
$$PE = PE_{1\&2} + PE_{2\&3} + PE_{1\&3} \quad (1)$$

$$= k_e \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right) \quad (2)$$

Here r_{12} is the distance between charge 1 and 2, and so on. Since we have an equilateral triangle, all distances are 1 m. Since all charges are equal in magnitude, we can simplify this quite a bit once we plug in what we know - we just need to keep track of the signs of the charges:

$$\begin{aligned} PE_{\text{total}} &= k_e \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right) \\ &= \left(9 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2} \right) \left[\frac{(10^{-9} \text{ C})(-10^{-9} \text{ C})}{1 \text{ m}} + \frac{(10^{-9} \text{ C})(10^{-9} \text{ C})}{1 \text{ m}} + \frac{(-10^{-9} \text{ C})(10^{-9} \text{ C})}{1 \text{ m}} \right] \\ &= \left(9 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2} \right) \left(\frac{10^{-18} \text{ C}^2}{\text{m}} \right) [-1 + 1 - 1] \\ &= (9 \times 10^9 \text{ N} \cdot \text{m}) [-1] \\ &\approx -9 \times 10^{-9} \text{ J} \end{aligned}$$

Again, we used the conversion $1 \text{ J} \equiv 1 \text{ N} \cdot \text{m}$.



2. A parallel plate capacitor has a capacitance C when there is vacuum between the plates. The gap between the plates is half filled with a dielectric with dielectric constant κ in two different ways, as shown below. Calculate the effective capacitance, in terms of C and κ , for both situations. *Hint: try breaking each situation up into two equivalent capacitors.*

(a) Dielectric parallel to the plates: $C_{\text{eff}} = \frac{2K}{1+K}C$.

It is easiest to think of this as two capacitors in series, both with half the plate spacing - one filled with dielectric, one with nothing. First, without *any* dielectric, we will say that the original capacitor has plate spacing d and plate area A . The capacitance is then:

$$C_0 = \frac{\epsilon_0 A}{d} \quad (3)$$

The upper half capacitor with dielectric then has a capacitance:

$$C_d = \frac{K\epsilon_0 A}{d/2} = \frac{2K\epsilon_0 A}{d} = 2KC_0 \quad (4)$$

The half capacitor without then has

$$C_{\text{none}} = \frac{\epsilon_0 A}{d/2} = \frac{2\epsilon_0 A}{d} = 2C_0 \quad (5)$$

Now we just add the two like capacitors in series:

$$\frac{1}{C_{\text{eff}}} = \frac{1}{2KC_0} + \frac{1}{2C_0} \quad (6)$$

$$C_{\text{eff}} = \frac{4KC_0^2}{2KC_0 + 2C_0} \quad (7)$$

$$= \frac{2K}{1+K}C_0 \quad (8)$$

(b) Dielectric “perpendicular” to the plates: $C_{\text{eff}} = \frac{K+1}{2}C$.

In this case, we think of the half-filled capacitor as two capacitors in *parallel*, one filled with dielectric, one with nothing. Now each half capacitor has half the plate *area*, but the same spacing. The upper half capacitor with dielectric then has a capacitance:

$$C_d = \frac{K\epsilon_0 \frac{1}{2}A}{d} = \frac{K\epsilon_0 A}{2d} = \frac{1}{2}KC_0 \quad (9)$$

The half capacitor without then has

$$C_{\text{nonc}} = \frac{\epsilon_0 \frac{1}{2} A}{d} = \frac{\epsilon_0 A}{2d} = \frac{1}{2} C_0 \quad (10)$$

Now we just add our parallel capacitors:

$$C_{\text{eff}} = \frac{1}{2} K C_0 + \frac{1}{2} C_0 \quad (11)$$

$$= \frac{1}{2} (K + 1) C_0 \quad (12)$$

$$= \frac{K + 1}{2} C_0 \quad (13)$$

3. An electron has a velocity of 3×10^6 m/s perpendicular to a magnetic field and is observed to move in a circle of radius 0.3 m.

(a) What is the strength of the B field?

(b) What E field could you apply (in addition to the B field) to cause the electron to move in a straight line instead? Give the magnitude and direction (relative to the B field and the electron's velocity).

When a charged particle moves in a plane perpendicular to a magnetic field, circular motion results. The magnetic force must provide the centripetal force:

$$F_B = qv_{\perp} B = \frac{mv_{\perp}^2}{r}$$

$$B = \frac{mv_{\perp}}{qr} \approx 5.7 \times 10^{-5} \text{ T} = 57 \mu\text{T}$$

In order for the particle to travel in a straight line, we need the electric force qE to balance the magnetic force, resulting in zero net force.

$$F_B = F_e$$

$$qv_{\perp} B = qE$$

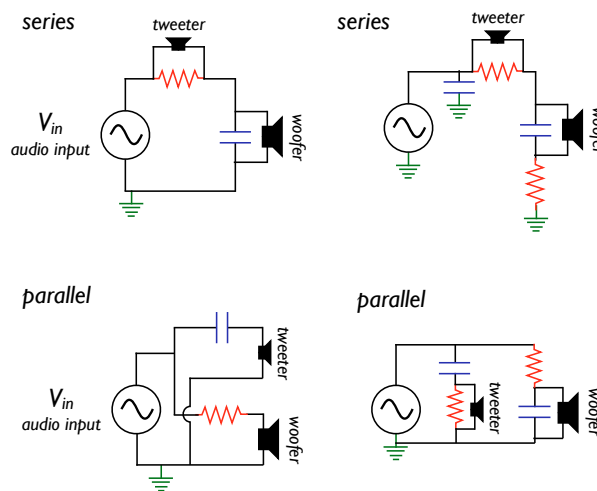
$$E = vB \approx 170 \text{ N/C}$$

4. Using capacitors, resistors, and inductors, sketch a circuit to split an audio signal composed of many frequencies in to a low frequency part and a high frequency part, for distribution to speakers. That is, filter the incoming signal into separate low frequencies and high frequencies to send to a woofer and tweeter, respectively. You do not need to specify the values of your components.

Even with only passive components like capacitors, inductors, and resistors, there are many ways to go about this problem. For simplicity, we will use only resistors and capacitors. We know that capacitors allow higher frequency signals through easily, while blocking lower frequency signals. A resistor,

on the other hand, lets all frequencies through equally. What we can do, then, is use a capacitor to direct high frequency signals away from the woofer and toward the tweeter.

The circuit in the upper left portion of the circuit below shows the simplest possible crossover. If you look closely, it is identical to the low-pass filter in the next problem! All we have done is connect the output - which will be preferentially low-frequency signals - to the woofer. The tweeter takes the full range signal developed across the resistor, which will include the high frequencies. Another way of thinking of this circuit is that the capacitor “shorts out” the high frequency signals, and keeps them away from the woofer. One serious way that this circuit is lacking is that we don’t send *only* high frequencies to the tweeter.

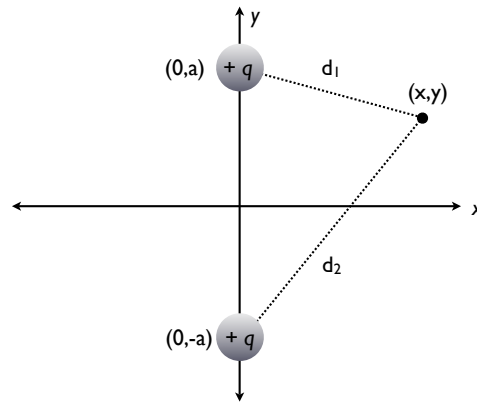


We could cure that problem by putting a low-pass and high-pass filter in series instead of just a resistor and capacitor, as shown in the upper right panel in the figure above. The low pass filter connects across the tweeter, and shorts the low frequency signals around it, such that it sees only high frequencies. The high pass filter does the reverse for the woofer, so it only sees low frequency signals. This is a perfectly workable crossover, and on paper it does just what we want. A major disadvantage (or, an advantage depending on your viewpoint) is that the two filters interact with each other - each filter is sending some signal to ground that really should be going to the other. Further, changes in any one component affect *both* high and low pass filter sections - to keep the same overall flat response, one must change all components, not just one, when one tries to tune the crossover frequency. Finally, this circuit is very sensitive to component variations and (in)accuracy. Still, the design is sufficiently simple to be appealing.

How can we do better? We can make a voltage divider to split the input signal into two, and send half to each filter and speaker. This is a *parallel* crossover. Rather than make a voltage divider out of resistors, we make one out of a pair of filters. In the lower left panel of the figure above, the audio input signal is split into two. On the branch going to the tweeter, we put a capacitor in series, which ensures that only the high-frequency part of the signal will take this path to the tweeter. The leftover low-frequency parts of the signal take the other branch to the woofer. This design is much more common, mainly because the two filter sections do not interact, which means they can be designed separately. Further, the sensitivity to component variation is far less. The lower right diagram is a re-rendering of the same circuit, adding a shorting capacitor and resistor for the woofer and tweeter. In this geometry, it is (perhaps) easier to see that both filters receive the same signal, and thus act independently.

5. (a) Two identical point charges $+q$ are located on the y axis at $y = +a$ and $y = -a$. What is the electric potential for an arbitrary point (x, y) ?

$V = \frac{k_e q}{\sqrt{x^2 + (a-y)^2}} + \frac{k_e q}{\sqrt{x^2 + (a+y)^2}}$. For this one, it is perhaps easier to draw ourselves a picture:



We will label the upper charge 1, and the lower charge 2. The principle of superposition tells us that we only need to find the potential at point (x, y) due to each separately, and then add the results together. First, we focus on charge 1, located at $(0, a)$. First, we need the distance d_1 from charge 1 to the point (x, y) . The horizontal distance is just x , and the vertical distance has to be $a - y$. Therefore,

$$d_1 = \sqrt{x^2 + (a - y)^2} \quad (14)$$

The potential due the first charge, which we'll call V_1 is then found easily:

$$V_1 = \frac{k_e q}{d_1} = \frac{k_e q}{\sqrt{x^2 + (a - y)^2}} \quad (15)$$

The potential due to the second charge at $(0, -a)$ is found in an identical manner, only noting that the vertical distance is now $a + y$:

$$d_2 = \sqrt{x^2 + (a + y)^2} \quad (16)$$

$$V_2 = \frac{k_e q}{d_2} = \frac{k_e q}{\sqrt{x^2 + (a + y)^2}} \quad (17)$$

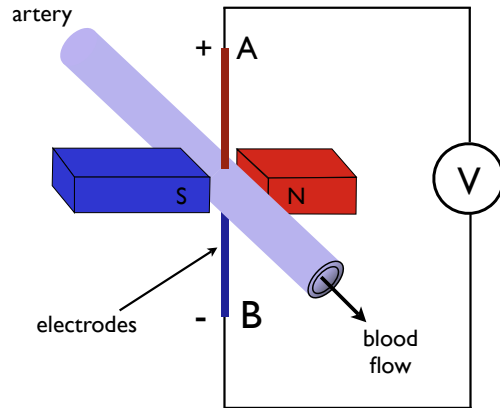
Finally, since potential is a scalar quantity (it has only magnitude, not direction), the superposition principle tells us that the total electric potential at point (x, y) is just the sum of the individual poten-

tials due to charges 1 and 2:

$$V_{\text{tot}} = V_1 + V_2 = \frac{k_e q}{\sqrt{x^2 + (a - y)^2}} + \frac{k_e q}{\sqrt{x^2 + (a + y)^2}} \quad (18)$$

Without resorting to approximations, there isn't really a much more aesthetically pleasing form for this one.

6. Using an electromagnetic flowmeter (see figure), a heart surgeon monitors the flow rate of blood through an artery. Electrodes A and B make contact with the outer surface of the blood vessel, which has inside **diameter** 3.2 mm. Permanent magnets outside the blood vessel create a magnetic field perpendicular to the blood flow direction. For a magnetic field strength of $|\vec{B}| = 0.037 \text{ T}$, a potential difference of $\Delta V = 160 \mu\text{V}$ appears between the electrodes.



- (a) Calculate the speed of the blood.
 (b) Does the sign of the potential difference depend on whether the mobile ions in the blood are predominantly positively or negatively charged?

1.35 m/s. The presence of a magnetic field perpendicular to the movement of ions in the blood means that they experience a magnetic force.

$$F_m = qvB$$

The force for positive ions is pointing up, and for negative ions it is pointing down. This serves to separate spatially the positive and negative charges - positive charges move toward electrode A, and negative charges move to point B. This continues until the ions reach the surface of the artery, at which point they are separated by the diameter of the artery d .

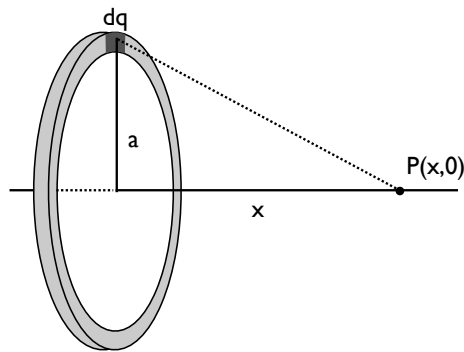
If the charges are separated spatially by a distance d , then this gives rise to an electric field E , and a potential difference $\Delta V = Ed$. At this point, the electric and magnetic forces are balanced. Set the magnetic and electric fields equal to one another, use the expression for ΔV , and take care with units:

$$\begin{aligned} F_m &= qvB = F_e = qE = q\left(\frac{\Delta V}{d}\right) \\ \implies v &= \frac{q\Delta V}{qBd} = \frac{\Delta V}{Bd} = 1.35 \text{ m/s} \end{aligned}$$

You can verify that the sign of the potential difference does *not* depend on whether the ions are mostly positive or negative, since the force is in different directions for each ion. No matter what, positive ions go to point A, and negative ions go to point B, and the same potential difference results.

7. A circular ring of charge of radius a has a total positive charge Q distributed uniformly around it. The ring is in the $x = 0$ plane with its center at the origin. What is the electric field (both magnitude and direction) along the x axis at an arbitrary point $x = b$ due to the ring of charge? *Hint: Consider the total charge Q to be made up of many pairs of identical charges placed on opposite points on the ring.*

We will find both the potential and the field, just for fun. Use the figure below to set up the problem. The ring is oriented with its plane perpendicular to the x axis, with its center at the origin. Since the ring is a continuous distribution of charge, we will have to integrate the electric field contributions from infinitesimal segments of the ring to find the total field.



Let our point of interest be P along the x axis. The distance from an infinitesimal element of the ring dq to P is $\sqrt{x^2 + a^2}$, so the electric potential from our element dq at P must be

$$dV = \frac{k_e dq}{\sqrt{x^2 + a^2}}$$

The entire potential at P is just the result of integrating over all possible dq :

$$V = \int_{\text{ring}} \frac{k_e dq}{\sqrt{x^2 + a^2}}$$

Note, however, that every element dq is the same distance from P - x and a do not change, so our integral becomes a trivial one, $\int dq$, which is just the total charge on the ring Q .

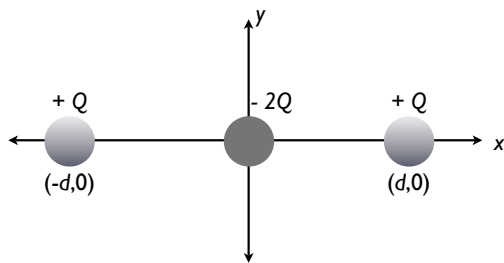
$$V = \frac{k_e}{\sqrt{x^2 + a^2}} \int_{\text{ring}} dq = \frac{k_e Q}{\sqrt{x^2 + a^2}}$$

The electric field is most easily found by noting $E = -\vec{\nabla}V$, or since we have only an x coordinate to worry about, $\vec{E} = -dV/dx \hat{x}$:

$$\vec{E} = -\frac{dV}{dx} \hat{x} = -\frac{d}{dx} \frac{k_e Q}{\sqrt{x^2 + a^2}} \hat{x} = -(2x) \left(\frac{1}{2}\right) \frac{k_e Q}{(x^2 + a^2)^{3/2}} \hat{x}$$

$$\vec{E} = \frac{k_e Q x}{(x^2 + a^2)^{3/2}} \hat{x}$$

Of course, we could have just found \vec{E} directly by integration, which is an example problem in Ch. 23 of your text. Many times, however, it is easier to find the potential first (since this does not require fiddling with vectors) and then take its gradient.



8. The charge distribution shown is referred to as a linear quadrupole. (a) What is the electric potential at a point on the x axis, where $x > d$? (b) What is the electric field for an *arbitrary* point (x, y) ?

It is marginally easier in the end to find the electric potential at an arbitrary point (x, y) , then find the field from $\vec{E} = -\vec{\nabla}V$. First, write down the distance to each charge for an arbitrary point (x, y) . Let the charges be numbered 1, 2, and 3 from left to right. Then we have, using the distance formula,

$$d_1^2 = (x + d)^2 + y^2$$

$$d_2^2 = x^2 + y^2$$

$$d_3^2 = (x - d)^2 + y^2$$

Note that this does work for any point, if we are careful about the sign of x . The potential at (x, y) is then just

$$V(x, y) = \frac{k_e Q}{\sqrt{(x + d)^2 + y^2}} + \frac{-2k_e Q}{\sqrt{x^2 + y^2}} + \frac{k_e Q}{\sqrt{(x - d)^2 + y^2}}$$

At a point along the x axis, where $x > d$, we just need to set $y = 0$:

$$V(x, 0) = \frac{k_e Q}{x + d} + \frac{-2k_e Q}{x} + \frac{k_e Q}{x - d} = \frac{2k_e Q d^2}{x(x^2 - d^2)}$$

Note that for large $x \gg d$, we have $V \propto 1/x^3$, in contrast to a point charge ($V \propto 1/x$) or a dipole

$(V \propto 1/x^2)$.

Finding the field at an arbitrary point (x, y) just means we need to take the gradient of $V(x, y)$, which means taking a derivative for each vector component. We don't have any \hat{z} component in this two dimensional problem, which makes things a little easier.

$$\begin{aligned}\vec{\mathbf{E}}(x, y) &= -\vec{\nabla}V(x, y) = -\vec{\nabla} \left[\frac{k_e Q}{\sqrt{(x+d)^2 + y^2}} + \frac{-2k_e Q}{\sqrt{x^2 + y^2}} + \frac{k_e Q}{\sqrt{(x-d)^2 + y^2}} \right] \\ \Rightarrow E_x &= -\frac{d}{dx} \left[\frac{k_e Q}{\sqrt{(x+d)^2 + y^2}} + \frac{-2k_e Q}{\sqrt{x^2 + y^2}} + \frac{k_e Q}{\sqrt{(x-d)^2 + y^2}} \right] \hat{\mathbf{x}} \\ E_y &= -\frac{d}{dy} \left[\frac{k_e Q}{\sqrt{(x+d)^2 + y^2}} + \frac{-2k_e Q}{\sqrt{x^2 + y^2}} + \frac{k_e Q}{\sqrt{(x-d)^2 + y^2}} \right] \hat{\mathbf{y}}\end{aligned}$$

It looks a bit messy, but it isn't so bad once you get started ...

$$\begin{aligned}E_x &= -\frac{d}{dx} \left[\frac{k_e Q}{\sqrt{(x+d)^2 + y^2}} + \frac{-2k_e Q}{\sqrt{x^2 + y^2}} + \frac{k_e Q}{\sqrt{(x-d)^2 + y^2}} \right] \hat{\mathbf{x}} \\ E_x &= \left[\frac{k_e Q (x-d)}{[(x-d)^2 + y^2]^{3/2}} + \frac{-2k_e Q x}{[x^2 + y^2]^{3/2}} + \frac{k_e Q (x+d)}{[(x+d)^2 + y^2]^{3/2}} \right] \hat{\mathbf{x}} \\ E_y &= -\frac{d}{dy} \left[\frac{k_e Q}{\sqrt{(x+d)^2 + y^2}} + \frac{-2k_e Q}{\sqrt{x^2 + y^2}} + \frac{k_e Q}{\sqrt{(x-d)^2 + y^2}} \right] \hat{\mathbf{y}} \\ E_y &= \left[\frac{k_e Q y}{[(x-d)^2 + y^2]^{3/2}} + \frac{-2k_e Q y}{[x^2 + y^2]^{3/2}} + \frac{k_e Q y}{[(x+d)^2 + y^2]^{3/2}} \right] \hat{\mathbf{y}}\end{aligned}$$

That's basically it, noting that $\vec{\mathbf{E}} = E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}}$. Without making approximations, there isn't a much more aesthetically-pleasing form.

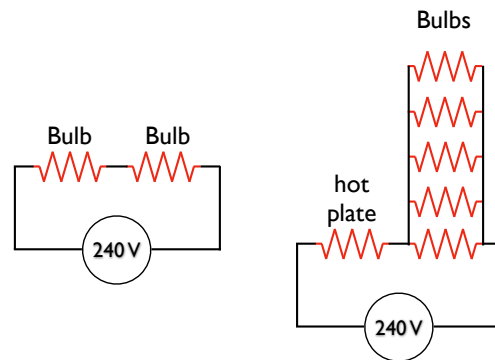
9. A group of students on spring break manages to reach a deserted island in their wrecked sailboat. They splash ashore with fuel, a European gasoline-powered 240 V generator, a box of North American 100 W, 120 V lightbulbs, a 500 W 120 V hot pot, lamp sockets, and some insulated wire. While waiting to be rescued they decide to use the generator to operate some bulbs.

(a) Draw a diagram of a circuit they can use, containing the minimum number of lightbulbs with 120 V across each bulb, and no higher output.

(b) One student catches a fish and wants to cook it in the hot pot. Draw a diagram of a circuit containing the hot pot and the minimum number of lightbulbs with 120 V across each device, and not more. Find

the current in the generator and its power output.

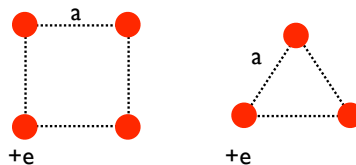
Below is a sketch of the simplest configurations in each case, you should be able to verify that both work (neither power nor voltage rating is exceeded for any component), and no components could be removed.



Briefly: for the first one, you want to make sure that the bulbs each get 120 V, or half of what the generator provides. Put two in series, and they each have half the voltage. Is the power rating OK? Given a power rating and a voltage rating for the bulbs, you can calculate both the resistance and maximum allowed current. Given the resistance of the bulb, you can calculate the current in the left-hand diagram above, and verify that each bulb receives exactly 100 W.

For the second one, simply putting the bulb and hot pot in series is no good - they don't have the same resistance, so they won't split the voltage equally anymore. One (the bulb) will take too much voltage and bad things will happen. Thus, we need to reduce the resistance of a set of bulbs before we can put them in series with the hot pot and divide the voltage between them. We can reduce the resistance of a set of bulbs by putting them in parallel with each other. You can verify that 5 bulbs in parallel have the same resistance as the hot pot, so they will all have 120 V as will the hot pot in the right-hand configuration. Further, given the resistance of each, you can calculate the total current, total power, and current in each component and verify that nothing has exceeded its rated power.

10. A single atomic layer of singly-charged ions (charge $+e$) can be arranged on a neutral insulating surface in one of two ways: either as a square or a triangular lattice. For four ions in the former configuration, and three in the latter, calculate the potential energy *per unit charge*. Which lattice is more stable?



The potential energy of the whole lattice is found by summing up the potential energies of every pair of charges in the system. For the square lattice, we have six unique pairs: four between adjacent

corners (distance a), and two between opposite corners (distance $a\sqrt{2}$). Number the charges 1-4 clockwise around the square.

$$\begin{aligned}
 PE_{\square} &= \sum_{\substack{\text{unique pairs} \\ ij}} \frac{k_e q_i q_j}{r_{ij}} \\
 &= \frac{k_e q_1 q_2}{a} + \frac{k_e q_2 q_3}{a} + \frac{k_e q_3 q_4}{a} + \frac{k_e q_4 q_1}{a} + \frac{k_e q_1 q_3}{a\sqrt{2}} + \frac{k_e q_2 q_4}{a\sqrt{2}} \\
 &= 4 \cdot \frac{k_e e^2}{a} + 2 \cdot \frac{k_e e^2}{a\sqrt{2}} \\
 &= \frac{k_e e^2}{a} \left[4 + \frac{2}{\sqrt{2}} \right] \\
 &= \frac{k_e e^2}{a} [4 + \sqrt{2}]
 \end{aligned}$$

Since there are four charges, the potential energy per unit charge is

$$PE_{\square}/\text{charge} = \frac{k_e e^2}{a} \left[1 + \frac{\sqrt{2}}{4} \right] \approx 1.35 \frac{k_e e^2}{a}$$

For the triangular lattice, we proceed in the same way, but now there are only three possible pairings of charges (1-2, 2-3, 1-3), and they are all the same distance apart:

$$\begin{aligned}
 PE_{\triangle} &= \sum_{\substack{\text{unique pairs} \\ ij}} \frac{k_e q_i q_j}{r_{ij}} \\
 &= \frac{k_e q_1 q_2}{a} + \frac{k_e q_2 q_3}{a} + \frac{k_e q_3 q_1}{a} \\
 &= 3 \frac{k_e e^2}{a}
 \end{aligned}$$

Since there are three charges, the potential energy per unit charge is just

$$PE_{\triangle}/\text{charge} = \frac{k_e e^2}{a}$$

This energy is smaller than that of the square lattice, which means that the triangular lattice is the more stable one. Neither is truly *stable* though - since the energy is positive, it is more favorable to not have the lattice at all, but separate the charges. This is not unexpected, however, since all charges are positive.

11. Three capacitors of 2, 4, and 6 μF , respectively, are connected in series, and a potential difference of 200 V is established across the whole combination by connecting the free terminals to the battery. **(a)** Calculate the charge on each capacitor. **(b)** Find the potential difference across each capacitor.

Capacitors in series all have the same *charge*, and they have the same charge as the equivalent capacitor. For series capacitors, we know how to find the equivalent capacitance:

$$\frac{1}{C_{eq}} = \frac{1}{2 \mu\text{F}} + \frac{1}{4 \mu\text{F}} + \frac{1}{6 \mu\text{F}} = \frac{11}{12 \mu\text{F}}$$

$$\Rightarrow C_{eq} = \frac{12}{11} \mu\text{F}$$

The total charge on all capacitors is just the charge on the equivalent capacitor:

$$Q_{\text{tot}} = C_{eq} \Delta V = \left(\frac{12}{11} \mu\text{F} \right) (200 \text{ V}) \approx 218 \mu\text{C}$$

Now: remember that for series capacitors, *each capacitor has the same charge, which is the same as the charge on the equivalent capacitor*. Thus, every capacitor carries $218 \mu\text{C}$. We can find the voltage on each one easily, knowing that:

$$\Delta V_2 = \frac{Q_{\text{tot}}}{2 \mu\text{F}} = 109.1 \text{ V}$$

$$\Delta V_4 = \frac{Q_{\text{tot}}}{4 \mu\text{F}} = 54.5 \text{ V}$$

$$\Delta V_6 = \frac{Q_{\text{tot}}}{6 \mu\text{F}} = 36.4 \text{ V}$$

As a double check, for series components the voltage drops should add up to the battery voltage:

$$\Delta V_{\text{tot}} = \Delta V_2 + \Delta V_4 + \Delta V_6 = 200 \text{ V}$$

12. Three equal positive charges, each of magnitude Q , are held fixed at the corners of a square of side a .
(a) Find the magnitude and direction of the electric field at the fourth corner. **(b)** Find the potential at the fourth corner. **(c)** How much work would be done in moving a fourth charge q to the fourth corner.

Let the origin be on the fourth corner, with the x and y axes aligned along the square's sides – this puts one charge at $(-a, 0)$, one at $(0, a)$, and one at $(-a, a)$. The charge at $(-a, 0)$ gives an electric field component in only the \hat{x} direction, and the one at $(0, a)$ in only the \hat{y} direction. The charge at $(-a, a)$ gives a field at 45° to either axis. The total field can be found by adding up the individual components:

$$E_x = E_{(0,a)} + E_{(-a,a)} \cos 45^\circ = \frac{k_e Q}{a^2} + \frac{k_e Q \sqrt{2}}{2a^2} \frac{\sqrt{2}}{2} = \frac{k_e Q}{a^2} \left(1 + \frac{\sqrt{2}}{4} \right)$$

Based on the symmetry of the problem, the y component must be the same (except that it points downward, and acquires a minus sign):

$$E_y = -\frac{k_e Q}{a^2} \left(1 + \frac{\sqrt{2}}{4} \right)$$

The total field is then just:

$$\vec{\mathbf{E}} = \left[\frac{k_e Q}{a^2} \left(1 + \frac{\sqrt{2}}{4} \right) \right] \hat{\mathbf{x}} - \left[\frac{k_e Q}{a^2} \left(1 + \frac{\sqrt{2}}{4} \right) \right] \hat{\mathbf{y}} = \left[\frac{k_e Q}{a^2} \left(1 + \frac{\sqrt{2}}{4} \right) \right] [\hat{\mathbf{x}} - \hat{\mathbf{y}}]$$

The potential is more easily found, since we just need the distances of the three charges:

$$V_{\text{tot}} = \frac{k_e Q}{a} + \frac{k_e Q}{a} + \frac{k_e Q}{a\sqrt{2}} = \frac{k_e Q}{a} \left(2 + \frac{1}{\sqrt{2}} \right)$$

Could we have found the potential first, and then taken the gradient? Yes, with the caveat that we put the charges at arbitrary positions $(-x, 0)$, $(0, y)$, and (x, y) ... one cannot take the gradient if only constants are involved.

How much work is done in moving a charge to the fourth corner? That is just the magnitude of the fourth charge time the electric potential at the corner, if we take the zero of potential to be an infinite distance away. Remember, adding a charge is the same as bringing it in from an infinite distance away. If the potential is zero an infinite distance away, we have to move the new charge through a potential difference of $\Delta V = V_{\text{tot}}$, so the work is just qV_{tot}

$$W = q\Delta V = qV_{\text{tot}} = \frac{k_e qQ}{a} \left(2 + \frac{1}{\sqrt{2}} \right)$$

13. You are given a voltage source, several very precise resistors, and two very accurate voltmeters. The voltmeter has an internal resistance of $100 \text{ M}\Omega$. Design a circuit to determine accurately the resistance of an unknown specimen, and estimate an upper limit on the specimen's resistance which will maintain 5% accuracy.

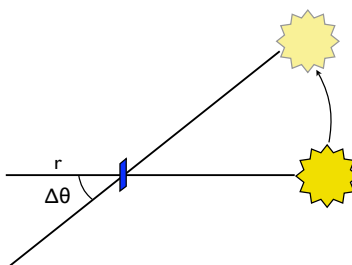
14. The walls of a prison cell are perpendicular to the four cardinal compass directions. On the first day of spring, light from the rising Sun enters a rectangular window in the eastern wall. The light traverses 2.57 m horizontally to shine perpendicularly on the wall opposite the window. A prisoner observes the patch of light moving across this western wall and for the first time forms his own understanding of the rotation of the Earth. (a) With what speed does the illuminated rectangle move? (b) The prisoner holds a small square mirror flat against the wall at one corner of the rectangle of light. The mirror reflects light back to a spot on the eastern wall close beside the window. How fast does the smaller square of light move across that wall?

The sun appears to move at an angular velocity ω , which means that it moves through an angular displacement $\Delta\theta$ in a time Δt : $\omega = \Delta\theta / \Delta t$. We know the rotation rate of the sun: it goes through a full circle of 2π radians in 24 hours:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi \text{ rad}}{86400 \text{ s}} \approx 7.27 \times 10^{-5} \text{ rad/s}$$

The light streaming through the prison window will move through an angle $\Delta\theta$ as shown below as

the sun moves through the sky:



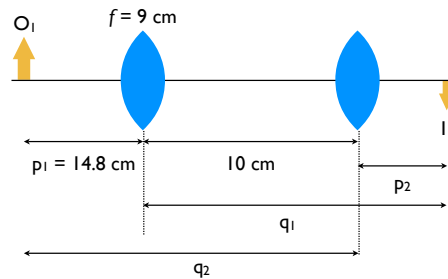
If the distance the light covers along the wall is s , then it is clear that $s = \Delta\theta r$. The rate at which the spot moves is ds/dt . Since $r = 2.37$ m is constant,

$$\frac{ds}{dt} = \frac{d}{dt} (\Delta\theta r) = r \frac{d\Delta\theta}{dt} = r\omega = (2.37 \text{ m}) (7.27 \times 10^{-5} \text{ rad/s}) \approx 0.172 \text{ mm/s}$$

If the prisoner uses a mirror, the path length of the light is simply doubled, as if the room were twice as wide, so a given angular displacement $\Delta\theta$ results in twice as large a lateral displacement s , and twice the apparent speed ds/dt . Thus, for the second case, we have just 0.345 mm/s.

15. Object O_1 is 14.8 cm to the left of a converging lens with a 9.0 cm focal length. A second lens is positioned 10.0 cm to the right of the first lens and is observed to form a final image at the position of the original object, O_1 . (a) What is the focal length of the second lens? (b) What is the overall magnification of this system? (c) What is the nature (i.e., real or virtual, upright or inverted) of the final image?

Object O_1 is outside the focal length of the first lens, so it must form a real, inverted image. This image I_1 serves as an object for the second lens. If this image, serving as an object for the second lens, is to form an image back at the original position of O_1 , where must I_1 be located? It cannot be in the region between the two lenses, or it would form an image to the right of the second lens. It must be to the *right* of the second lens, and outside its focal point, as shown below:



Inverted image I_1 is then imaged by the second lens to form a virtual, upright image back at the position of O_1 (the image is virtual because no real light rays go back to the left). This makes sense: if I_1 were not to the right of the second lens, and outside its focal point, there is no way that the final image could be in the same place as O_1 . Using the distances labeled in the figure, we note the following relations between the distances from the first and second lenses:

$$p_2 = q_1 - 10$$

$$q_2 = 10 + p_1$$

Forming image I_1 from O_1 with the first lens, we can say

$$\frac{1}{f_1} = \frac{1}{p_1} + \frac{1}{q_1}$$

$$\frac{1}{q_1} = \frac{1}{f_1} - \frac{1}{p_1} = \frac{1}{9} - \frac{1}{14.8}$$

$$\Rightarrow q_1 \approx 23.0 \text{ cm}$$

Forming the final image from I_1 with the second lens, we have

$$\frac{1}{f_2} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{q_1 - 10} + \frac{1}{p_1 + 10} = \frac{1}{13} + \frac{1}{24.8}$$

$$\Rightarrow f_2 \approx 8.51 \text{ cm}$$

How about the magnification? The magnification factor M of two lenses in "series" is just the product of their individual magnification factors M_1 and M_2 respectively: $M = M_1 M_2$. Basically: O_1 is

magnified by a factor $M_1 = -q_1/p_1$ to form I_1 , which is then magnified by a factor $M_2 = -q_2/p_2$ to form the final image. If both lenses, say, reduce the image by 50%, the final image is 25% as large as the original object. Anyway:

$$M = M_1 M_2 = \frac{q_1 q_2}{p_1 p_2} = \frac{q_1 (p_1 + 10)}{p_1 (q_1 - 10)} \approx 2.97$$

Since $M > 0$, the image is upright and magnified. We also know that the image is *virtual*, since it forms on the “wrong” side of the lenses, and cannot be formed by real light rays. Thus the image is **virtual, upright, and magnified**.

16. A dedicated sports car enthusiast polishes the inside and outside surfaces of a hubcap that is a section of a sphere. When he looks into one side of the hubcap, he sees an image of his face 28.0 cm in back of the hubcap. He then turns the hubcap over, keeping it the same distance from his face. He now sees an image of his face 10.8 cm in back of it. **(a)** How far is his face from the hubcap? **(b)** What is the radius of curvature of the hubcap?

We will assume that the distance from the enthusiast’s face to the hubcap remains the same in both cases. One side of the hubcap is a convex spherical mirror, the other a concave spherical mirror. The image *behind* the hubcap, the closer one, must come from using the convex side of the hubcap. All we need to do now is use the mirror equation: for the concave side, the radius of curvature is R , while for the convex side, it is $-R$. In both cases, $q < 0$, since both images are behind the mirror.

Concave side: $q = -28$ cm,

$$\frac{1}{p} - \frac{1}{28} = \frac{2}{R}$$

$$\frac{2}{R} = \frac{28 - p}{28p}$$

Convex side: $q = -10.8$,

$$\frac{1}{p} - \frac{1}{10.8} = \frac{-2}{R}$$

$$\frac{2}{R} = \frac{p - 10.8}{10.8p}$$

We can use the two equations above to solve for p , the distance from the enthusiast’s face to the hubcap:

$$\frac{28 - p}{28p} = \frac{p - 10.8}{10.8p}$$

$$10.8p(28 - p) = 28p(p - 10.8) \quad (p \neq 0)$$

$$10.8(28 - p) = 28(p - 10.8)$$

$$38.8p = 604.8$$

$$\implies p = 15.6 \text{ cm}$$

Given p , we can also find R using the first equation above:

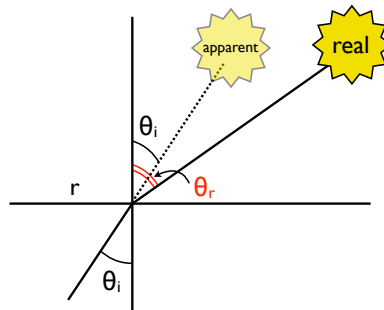
$$\frac{2}{R} = \frac{28 - p}{28p}$$

$$56p = 28R - pR$$

$$R = \frac{56p}{28 - p} \approx 70.3 \text{ cm}$$

17. A scuba diver is underwater; from there, the sun appears to be at 30° from the vertical. At what actual angle is the sun located (with respect to directly overhead)? The index of refraction of water may be taken as $n_{\text{water}} = 1.33$, while that of the air may be taken as $n_{\text{air}} = 1.00$.

As usual, we should first draw a picture. The swimmer looks out of the water at an angle θ_i , and light rays coming to the swimmer's eyes are bent to a larger angle θ_r at the water-air interface. Since the air represents a faster medium for light, the rays are bent *away* from the normal, so that $\theta_r > \theta_i$. The incident angle is the angle where the sun *appears* to be, the angle that the rays would make with the normal in the absence of refraction. The actual angle is θ_r .



All we need to do now is apply Snell's law:

$$n_{\text{water}} \sin \theta_i = n_{\text{air}} \sin \theta_r$$

$$1.33 \sin 30^\circ = \sin \theta_r$$

$$\implies \theta_r \approx 42^\circ$$