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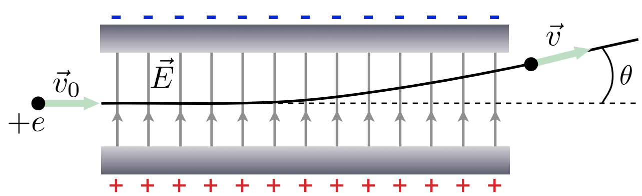
UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 106-4 / LeClair Fall 2008

Exercises

1. **10 points.** An ion milling machine uses a beam of gallium ions ($m = 70\,\mathrm{u}$) to carve microstructures from a target. A region of uniform electric field between parallel sheets of charge is used for precise control of the beam direction. Single ionized gallium atoms with initially horizontal velocity of 1.8×10^4 m/s enter a 2.0 cm-long region of uniform electric field which points vertically upward, as shown below. The ions are redirected by the field, and exit the region at the angle θ shown. If the field is set to a value of $E = 90\,\mathrm{N/C}$, what is the exit angle θ?



A singly-ionized gallium atom has a charge of $q = +e$, and the mass of $m = 70\,\mathrm{u}$ means 70 *atomic mass units*, where one atomic mass unit is $1\,\mathrm{u} = 1.66\times10^{-27}$ kg.

What we really have here is a particle under the influence of a constant force, just as if we were to throw a ball horizontally and watch its trajectory under the influence of gravity (the only difference is that since we have negative charges, things can "fall up"). To start with, we will place the origin at the ion's initial position, let the positive x axi run to the right, and let the positive y axis run straight up. Thus, the particle starts with a velocity purely in the x direction: $\vec{\mathbf{v}}_0 = v_x\,\hat{\mathbf{x}}$.

While the particle is in the electric-field-containing region, it will experience a force pointing along the $+y$ direction, with a constant magnitude of qE. Since the force acts only in the y direction, there will be a net acceleration only in the y direction, and *the velocity in the x direction will remain constant.* Once outside the region, the particle will experience no net force, and it will therefore continue along in a straight line. It will have acquired a y component to its velocity due to the electric force, but the x component will still be v_x. Thus, the particle exits the region with velocity $\vec{\mathbf{v}} = v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}}$. The angle at which the particle exits the plates, measured with respect to the x axis, must be

$$\tan\theta = \frac{v_y}{v_x}$$

Thus, just like in any mechanics problem, finding the angle is reduced to a problem of finding the final velocity components, of which we already know one. So, how do we find the final velocity in the y direction? Initially, there is no velocity in the y direction, and while the particle is traveling between the plates, there is a net force of qE in the y direction. Thus, the particle experiences an acceleration

$$a_y = \frac{F_y}{m} = \frac{qE_y}{m}$$

The electric field is purely in the y direction in this case, so $E_y = 90\,\mathrm{N/C}$. Now we know the acceleration in the y direction, so if we can find out the time the particle takes to transit the plates, we are done, since the the transit time Δt and acceleration a_y determine v_y:

$$v_y = a_y\Delta t$$

Since the x component of the velocity is not changing, we can find the transit time by noting that the distance covered in the x direction must be the x component of the velocity times the transit time. The distance covered in the x direction is just the width of the plates, so:

$$d_x = v_x\Delta t = 2.0\,\mathrm{cm} \quad\Longrightarrow\quad \Delta t = \frac{d_x}{v_x}$$

Putting the previous equations together, we can express v_y in terms of known quantities:

Name:

$$v_y = a_y \Delta t = a_y \frac{d_x}{v_x} = \frac{qE_y}{m}\frac{d_x}{v_x} = \frac{qE_y d_x}{mv_x}$$

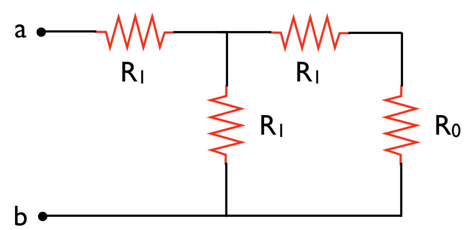
Finally, we can now find the angle θ as well:

$$\tan\theta = \frac{v_y}{v_x} = \frac{\frac{qE_y d_x}{mv_x}}{v_x} = \frac{qE_y d_x}{mv_x^2}$$

And that's that. Now we plug in the numbers we have, watching the units carefully:

$$\begin{aligned}
\theta &= \tan^{-1}\left[\frac{qE_y d_x}{mv_x^2}\right] \\
&= \tan^{-1}\left[\frac{\left(1.6\times10^{-19}\,\mathrm{C}\right)\left(90\,\mathrm{N/C}\right)\left(0.02\,\mathrm{m}\right)}{\left(70\cdot1.66\times10^{-27}\,\mathrm{kg}\right)\left(1.8\times10^{4}\,\mathrm{m/s}\right)^2}\right] \\
&= \tan^{-1}\left[7.6\times10^{-3}\,\frac{\mathrm{N}}{\mathrm{kg\cdot m/s^2}}\right] \qquad \text{note } 1\,\mathrm{N}{=}1\,\mathrm{kg\cdot m/s^2} \\
&= \tan^{-1} 7.6\times10^{-3} \\
&\approx 0.44^\circ
\end{aligned}$$

2. 15 points. In the circuit below, if R_0 is given, what value must the R_1 have for the equivalent resistance between the two terminals a and b to be R_0?



This one is, admittedly, a bit messy. The end result does have a certain elegance though ...

With any complicated resistor problem, we first try to find sets of two resistors purely in parallel or purely in series. Combine any such pairs, lather, rinse, repeat. The first pair we can spot - and the only one which is purely in series or parallel - is resistor R_0 in series with the rightmost R_1. We cannot combine any other resistors, since no other pairs are purely in series or parallel. Putting together R_1 and R_0 makes an equivalent resistor R_2, whose value we can calculate easily:

$$R_2 = R_1 + R_0$$

This will leave the new resistor purely in *parallel* with the middle R_1, which means we can combine R_2 and R_1 into a new resistor R_3:

$$\begin{aligned}
\frac{1}{R_3} &= \frac{1}{R_2} + \frac{1}{R_1} = \frac{1}{R_1+R_0} + \frac{1}{R_1} = \frac{R_1+R_0+R_1}{R_1\left(R_1+R_0\right)} = \frac{2R_1+R_0}{R_1^2+R_1R_0} \\
\Longrightarrow \quad R_3 &= \frac{R_1R_0+R_1^2}{2R_1+R_0}
\end{aligned}$$

Our progress so far is shown below.
Now we only have R_3 and one R_1 left, purely in series. Combining them will give us one single equivalent resistor R_{eq}:

$$\begin{aligned}
R_{eq} = R_1 + R_3 &= \frac{R_1R_0+R_1^2}{2R_1+R_0} + R_1 = \frac{R_1R_0+R_1^2}{2R_1+R_0} + \frac{R_1\left(2R_1+R_0\right)}{2R_1+R_0} \\
&= \frac{R_1R_0+R_1^2+2R_1^2+R_1R_0}{2R_1+R_0} \\
&= \frac{3R_1^2+2R_1R_0}{2R_1+R_0}
\end{aligned}$$

The final bit of the problem says that we want the equivalent resistance to be exactly R_0. We just need to set the above equal to R_0, and solve for R_1 in terms of R_0.

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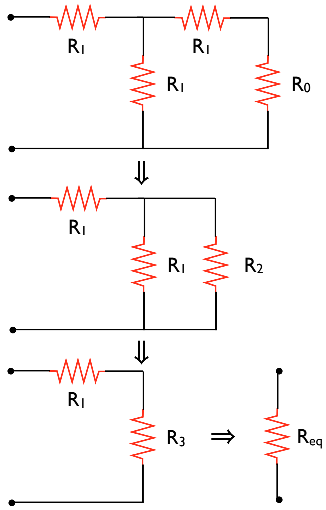


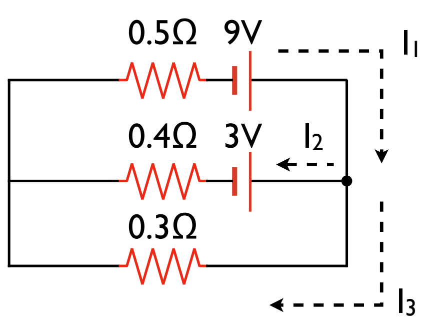
$$
\begin{aligned}
R_0 &= \frac{3R_1^2 + 2R_1R_0}{2R_1 + R_0} \\
R_0\left(2R_1 + R_0\right) &= 3R_1^2 + 2R_1R_0 \\
2R_0R_1 + R_0^2 &= 3R_1^2 + 2R_1R_0 \\
R_0^2 &= 3R_1^2 \\
\Longrightarrow \qquad R_1 &= \frac{R_0}{\sqrt{3}}
\end{aligned}
$$

3. 10 points. You are given two batteries, one of $9\,\mathrm{V}$ and internal resistance $0.50\,\Omega$, and another of $3\,\mathrm{V}$ and internal resistance $0.40\,\Omega$. How must these batteries be connected to give the largest possible current through an external $0.30\,\Omega$ resistor? What is this current?

There are basically two interesting ways to hook up the components given: all series, and all parallel. First, one can put everything in series. In series, the circuit is simple. You have three resistors and two batteries, and since there is only a single current in the circuit, which we'll call I, you can readily add up the voltage drops around the circuit to find I:

$$
\begin{aligned}
\text{series:} \quad -0.5\,\Omega\,I + 9\,\mathrm{V} - 0.4\,\Omega\,I + 3\,\mathrm{V} - 0.3\,\Omega\,I &= 0 \\
12\,\mathrm{V} - 1.2\,\Omega\,I &= 0 \\
I &= 10\,\mathrm{A}
\end{aligned}
$$

Putting everything in parallel looks like this:



In this case, there are three currents to deal with, it is the third I_3 that we are interested in. First, we can apply the "junction rule" at the circular dot on the right-hand side of the circuit. Current I_1 enters the junction, currents I_2 and I_3 leave:

$$I_1 = I_2 + I_3$$

Name:

Next, we can apply the "loop rule" around the upper-most loop, going clockwise. Remember that crossing a battery from the little pole (-) to the big pole (+) is a *gain* in voltage.

$$-0.5\,\Omega\, I_1 + 9\,\mathrm{V} - 3\,\mathrm{V} - 0.4\,\Omega\, I_2 = 0$$

We can do the same for the lower-most loop:

$$-0.4\,\Omega\, I_2 + 3\,\mathrm{V} - 0.3\,\Omega I_3 = 0$$

Summarizing our three equations so far (and dropping the units):

$$\begin{aligned} I_1 - I_2 - I_3 &= 0 \\ -0.5I_1 - 0.4I_2 &= -6 \\ 0.4I_2 - 0.3I_3 &= -3 \end{aligned}$$

We now have three equations and three unknowns. There are a few ways to go about solving them, I will illustrate two. First, plug the first equation into the third, and solve that for I_1

$$\begin{aligned} & 0.4I_2 - 0.3\,(I_1 - I_2) = 0.7I_2 - 0.3I_1 = -3 \\ \Longrightarrow \quad & I_1 = \frac{0.7}{0.3}I_2 + \frac{3}{0.3} \end{aligned}$$

Now plug that into the second equation we have:

$$\begin{aligned} -0.5I_1 - 0.4I_2 = -0.5\left(\frac{0.7}{0.3}\right) - 0.4\left(\frac{3}{0.3}\right) - 0.4I_2 = -6 \\ I_2\left(0.4 + 0.5\frac{0.7}{0.3}\right) = 6 - 0.5\left(\frac{3}{0.3}\right) \\ I_2 = 0.638\,\mathrm{A} \end{aligned}$$

Now that we have I_2, we can use the third equation to find I_3, the desired current through the $0.3\,\Omega$ resistor:

$$I_3 = \frac{0.4I_2 + 3}{0.3} = 10.85\,\mathrm{A}$$

Thus, connecting everything in parallel gives a slightly higher current through the resistor. One could also try to put two components in series and the third in parallel with that; you can quickly verify that none of those three combinations yield a larger current.

Another way to solve this, perhaps more quickly, is to use matrices and Cramer's rule,[i] if you are familiar with this technique. If you are not familiar with matrices, you can skip to the next problem - you are not required or necessarily expected to know how to do this. First, write the three equations in matrix form:

$$\begin{bmatrix} 1 & -1 & -1 \\ -0.5 & -0.4 & 0 \\ 0 & 0.4 & -0.3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ -3 \end{bmatrix}$$
$$\mathbf{aI} = \mathbf{V}$$

The matrix $\mathbf{a}$ times the column vector $\mathbf{I}$ gives the column vector $\mathbf{V}$, and we can use the determinant of the matrix $\mathbf{a}$ with Cramer's rule to find the currents. For each current, we construct a new matrix, which is the same as the matrix $\mathbf{a}$ except that the the corresponding column is replaced the column vector $\mathbf{V}$. Thus, for I_1, we replace column 1 in $\mathbf{a}$ with $\mathbf{V}$, and for I_2, we replace column 2 in $\mathbf{a}$ with $\mathbf{V}$. We find the current then by taking the new matrix, calculating its determinant, and dividing that by the determinant of $\mathbf{a}$. Below, we have highlighted the columns in $\mathbf{a}$ which have been replaced to make this more clear:

$$I_1 = \frac{\begin{vmatrix} 0 & -1 & -1 \\ -6 & -0.4 & 0 \\ -3 & 0.4 & -0.3 \end{vmatrix}}{\det \mathbf{a}} \qquad I_2 = \frac{\begin{vmatrix} 1 & 0 & -1 \\ -0.5 & -6 & 0 \\ 0 & -3 & -0.3 \end{vmatrix}}{\det \mathbf{a}} \qquad I_3 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ -0.5 & -0.4 & -6 \\ 0 & 0.4 & -3 \end{vmatrix}}{\det \mathbf{a}}$$

Now we need to calculate the determinant of each new matrix, and divide that by the determinant of $\mathbf{a}$.[ii] First, the determinant of $\mathbf{a}$.

[i] See 'Cramer's rule' in the Wikipedia to see how this works.
[ii] Again, the Wikipedia entry for 'determinant' is quite instructive.

Name: ______________________

$$\det a = (1)(-0.4)(-0.3) - (1)(0)(0.4) + (-1)(0)(0) - (-1)(-0.5)(-0.3) + (-1)(-0.5)(0.4) - (-1)(-0.4)(0) = 0.47$$

We can now find the currents readily from the determinants of the modified matrices above and that of $\mathbf{a}$ we just found. We really only want I_3, so we can find that directly:

$$I_3 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ -0.5 & -0.4 & -6 \\ 0 & 0.4 & -3 \end{vmatrix}}{\det \mathbf{a}} = \frac{3(0.4) + 6(0.4) + 3(0.5)}{0.47} = 10.85\,\text{A}$$

This time, we omitted the terms in the determinant which give zeros. Once you are familiar with this method of solving systems of equations, it can be quite efficient. You can complete the same procedure for I_2 and I_1, you should find $I_2 = 0.638\,\text{A}$ and $I_1 = 11.49\,\text{A}$.