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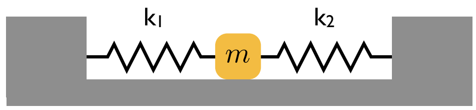
PH125 Exam III Solutions

1. A block of mass m is connected to two springs of force constants k_1 and k_2 as shown below. The block moves on a frictionless table after it is displaced from equilibrium and released. Determine the period of simple harmonic motion.



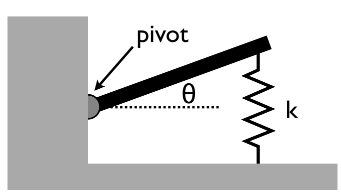
Let the $+x$ direction be to the right. If the mass is displaced by an amount Δx to the right, then the spring k_2 is compressed by Δx and k_1 must be expanded by Δx. Spring k_2 gives a force $-k_2\Delta x$ pushing the block back to the left, toward equilibrium, and spring k_2 gives a force $-k_1\Delta x$ pulling the block back toward equilibrium. The situation is the same if the mass is displaced to the left – both springs give a force acting in the same direction, toward the center, and both springs have the same displacement from equilibrium. The total force balance is thus

$$F_{\text{net}} = -k_1\Delta x - k_2\Delta x = (k_1+k_2)\,\Delta x = ma$$
$$a = -\left(\frac{k_1+k_2}{m}\right)x \equiv -\frac{k_{\text{eff}}}{m}x$$

This implies simple harmonic motion with period

$$T = 2\pi\sqrt{\frac{m}{k_{\text{eff}}}} = 2\pi\sqrt{\frac{m}{k_1+k_2}}$$

2. A horizontal plank of mass m and length L is pivoted at one end. The plank's other end is supported by a spring of force constant k. The moment of inertia of the plank about the pivot is $I=\frac{1}{3}mL^2$. The plank is displaced by an angle θ from horizontal equilibrium and released. Find the angular frequency ω of simple harmonic motion for small θ. *Hint:* start with the torques at equilibrium, and the spring in equilibrium compression.



The presence of a pivot point – even labeled as such – immediately suggests the use of torque to solve this problem. First, we need to find the compression of the spring at equilibrium, i.e., $\theta = 0$. Since the plank has non-zero mass, even without an angular displacement the spring must be compressed by some amount at equilibrium. Once we have found the equilibrium position, we can worry about the torques when a small angular displacement θ is applied.

Let counterclockwise rotations be defined as positive, and let the equilibrium position of the spring correspond to the tip of the plank being at vertical position x_o relative to its unstretched length. The sum of the torques about the pivot point at equilibrium ($\theta = 0$) is given by considering the weight of the plank acting about its center of mass and the restoring force of the spring. The plank may be treated as a point mass a distance $L/2$ from the pivot point, while the spring force acts at a distance L. At equilibrium, the net torque must be zero.

$$\sum \tau = -mg\left(\frac{L}{2}\right) + kx_oL = 0$$
$$x_o = \frac{mg}{2k}$$

Now that we have the equilibrium position, we can find the torques when the plank makes an angle θ with the vertical. Since the plank cannot change its length (we assume), the amount that the spring stretches should correspond to the arc length that the tip of the plank moves through, $L\theta$, if the angle is relatively small.[i] The spring is therefore displaced by an amount $L\theta - x_o$ from its unstretched length. This gives us the spring's restoring force. Since the spring is attached to the plank, the spring force always acts perpendicularly to the length of the plank at distance L, and the torque is easily found.

The plank's weight still acts at a distance $L/2$ from the center of mass, but now at an angle $90 - \theta$ relative to the axis of the plank. The overall torque is then

$$\sum \tau = -mg\left(\frac{L}{2}\right)\sin(90 - \theta) - k(L\theta - x_o)L = -\frac{1}{2}mgL\cos\theta - kL^2\theta + kx_oL$$

Since the angle θ is small, we may approximate $\cos\theta \approx 1$, and we may also make use of our earlier expression for x_o. Finally, out of equilibrium the torques must give the moment of inertia times the angular acceleration.

$$\sum \tau = -\frac{1}{2}mgL\cos\theta - kL^2\theta + kx_oL \approx -\frac{1}{2}mgL - kL^2\theta + \frac{1}{2}mgL = -kL^2\theta = I\alpha$$
$$-kL^2\theta = I\frac{d^2\theta}{dt^2}$$

Noting that $I = \frac{1}{3}mL^2$ for a thin plank, we can put the last equation in the desired form for simple harmonic motion in terms of known quantities:

[i] Small enough such that we don't have to worry about the spring bending to the left, for one.

$$\frac{d^2\theta}{dt^2} = -\frac{3k}{m}\theta^2$$
$$\omega = \sqrt{\frac{3k}{m}}$$

Note that the length of the plank does not matter at all.

3. Assume the earth to be a solid sphere of uniform density. A hole is drilled through the earth, passing through its center, and a ball is dropped into the hole. Neglect friction.

(a) Calculate the time for the ball to return to the release point. *Hint:* what sort of motion results?
(b) Compare the result of part a to the time required for the ball to complete a circular orbit of radius R_E about the earth

We consider a spherically-symmetric mass of uniform density, radius R_E, and mass M_E. If we are at a radius $r < R_E$ from the center of the earth, only the mass contained within $r < R_E$ gives rise to a net gravitational force – the gravitational interactions for bits of mass at $r > R_E$ all cancel each other. If we assume the mass of material removed to make the hole is negligible, then the mass contained within a radius r is simply

$$M(r) = M_E \left(\frac{r}{R_E}\right)^3$$

The force acting on a body of mass m at a distance r from earth's center is then

$$F(r) = -\frac{GM_E(r)m}{r^2} = -\frac{GM_Emr}{R_E^3}$$

Noting that the gravitational acceleration at earth's surface is

$$g = \frac{GM_E}{R_E^2}$$

we can write this compactly as

$$F(r) = -gm\frac{r}{R_E} = ma = m\frac{d^2r}{dt^2}$$
$$\frac{d^2r}{dt^2} = -\frac{g}{R_E}r$$

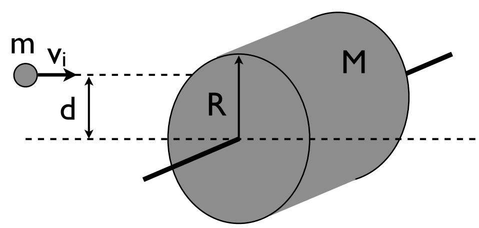
We have established that acceleration is a negative constant times position, which means that we have simple harmonic motion. Since the constant of proportionality between position and acceleration is g/R_E, we will have periodic motion with

$$T = 2\pi\sqrt{\frac{R_E}{g}}$$

Clearly, the time for the ball return to its original position is then T. We can compare this time to that required for an orbit around the earth at radius R_E (which would be an orbit skimming the surface of the earth, a bit silly ...) via Kepler's law, and they are the same:

$$T_{\text{orbit}} = 2\pi\sqrt{\frac{R_E^3}{GM_E}} = 2\pi\sqrt{\frac{R_E}{g}}$$

4. A wad of sticky clay with mass m and velocity v_i is fired at a solid cylinder of mass M and radius R as shown below. The cylinder is initially at rest and mounted on a fixed horizontal axle that runs through its center of mass. The line of motion of the projectile is perpendicular to the axis and at a distance $d < R$ from the center. Find the angular speed of the system just after the clay strikes and sticks to the surface of the cylinder. The moment of inertia of a solid cylinder is $I = \frac{1}{2}MR^2$, the moment of inertia of a point particle mass m a distance R from an axis of rotation is $I = mR^2$.



Clearly this is an inelastic collision, since the wad of clay sticks to the cylinder, and conservation of energy is right out. Conservation of momentum is fine, but the cylinder rotates after the collision and makes things difficult. Conservation of angular momentum will be a whole lot easier. Since we must eventually find the angular velocity of the cylinder, it is natural to consider angular momentum about the center of the cylinder.[ii]

Consider first the moment just before the wad hits the cylinder. The wad has momentum $p = mv$, acting at a distance R from the center of the cylinder. The angular momentum about the center of the cylinder is the wad's linear momentum times the perpendicular distance to the axis of rotation, which is just d. The angular momentum about the center of the cylinder is thus $L_i = mvd$.

After the wad hits the cylinder and both begin to rotate with angular velocity ω, the angular momentum can be found from the moments of inertia and ω. We have the wad, with $I_w = mR^2$, and the cylinder,

[ii] It does not really matter, angular momentum will be conserved regardless of our choice of origin. Choosing the center of the cylinder is simply more convenient.

with $I_c = \frac{1}{2}MR^2$, both rotating with angular velocity ω, and thus the total angular momentum after the collision is

$$L_f = I_w\omega I_c\omega = R^2\omega\left(m + \frac{1}{2}M\right)$$

Equating initial and final angular momentum, we can easily find ω:

$$R^2\omega\left(m + \frac{1}{2}M\right) = mvd$$

$$\omega = \frac{mvd}{R^2\left(m + \frac{1}{2}M\right)}$$

5. A satellite is in a circular Earth orbit of radius r. The area A enclosed by the orbit depends on r^2 because $A = \pi r^2$. Determine how the following properties depend on r: **(a)** period, **(b)** kinetic energy, **(c)** angular momentum, and **(d)** speed.

(a) The period is related to the orbital radius by Kepler's law:

$$T^2 = \left(\frac{4\pi^2}{GM}\right)r^3$$

$$T \propto r^{3/2}$$

(b) The gravitational potential energy of a satellite in an orbit of radius r is proportional to $1/r$. Conservation of energy dictates that any change in potential energy is made up by an equal change in kinetic energy, and thus the kinetic energy must also scale as $K \propto 1/r$.

(d) Velocity is related to kinetic energy, $v = \sqrt{2K/m}$, which means velocity must scale as $v \propto 1/\sqrt{r}$.

(c) Angular momentum depends on both velocity and radius, $L = mvr$, and thus we must have $L \propto \sqrt{r}$.

6. Here are some functions:

$$f_1(x,t) = Ae^{-b(x-vt)^2} \qquad f_2(x,t) = \frac{A}{b(x-vt)^2+1}$$

$$f_3(x,t) = Ae^{-b\left(bx^2+vt\right)} \qquad f_4(x,t) = A\sin(bx)\cos(bvt)^3$$

(a) Which ones satisfy the wave equation? Justify your answer with explicit calculations.
(b) For those functions that *satisfy* the wave equation, write down the corresponding functions $g(x,t)$ representing a wave of the same shape traveling in the opposite direction.

This is just grinding through the math, and showing that

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

It is seriously tedious. If you do this, you will find that f_1 and f_2 satisfy the wave equation, while f_3 and f_4 do not.

We found earlier that general solutions to the wave equation must have the form

$$h(x,t) = af(x+vt) + bg(x-vt)$$

where a and b are constants, and g and f represent forward- and backward-traveling wave solutions, respectively. Knowing this, the fact that only f_1 and f_2 are solutions is clear by inspection (though you were unfortunately asked to provide explicit calculations). Both f_1 and f_2 have the functional form $g(x-vt)$, meaning they are 'forward-traveling' solutions. The corresponding 'backward-traveling' solutions are found by simply changing the arguments to $x+vt$ from $x-vt$:

$$f_1'(x,t) = Ae^{-b(x+vt)^2} \qquad\qquad f_2'(x,t) = \frac{A}{b\left(x+vt\right)^2+1}$$

7. A string under tension T_i oscillates in the third harmonic ($n=3$) at frequency f_3, and the waves on the string have wavelength λ_3. If the tension is increased to $T_f = 4T_i$, and the string is again made to oscillate in the third harmonic, what are then the **(a)** frequency of oscillation in terms of f_3 and **(b)** the wavelength of the waves in terms of λ_3?

(a) We can relate the tension T and frequency f via the linear mass density of the wire μ, its length L, and the harmonic number n:

$$f = \frac{nv}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

Initially, the string is vibrating at its third harmonic ($n=3$) with tension T_i. Thus, the frequency of the third harmonic is

$$f_3 = \frac{3}{2L}\sqrt{\frac{T_i}{\mu}}$$

After increasing the tension to $T_f = 4T_i$, we are still at the third harmonic, which means *the wavelength must be the same*. Remember that the harmonic number tells us how many half-wavelenghts fit within the string's length L. The frequency is *not* the same now, but we can calculate it readily substituting the new tension:

$$f_3' = \frac{3}{2L}\sqrt{\frac{T_f}{\mu}} = \frac{3}{2L}\sqrt{\frac{4T_i}{\mu}} = 2f_3$$

If the tension increases by fourfold, the frequency for a given harmonic number doubles.

(b) The fact that the harmonic number is the same (with L being fixed) is enough to simply state that the wavelength must be unchanged, since the same number of half-wavelenghts fit within the same distance L. Just to be explicit, we can write the wavelength in terms of the harmonic number and the length of the wire:

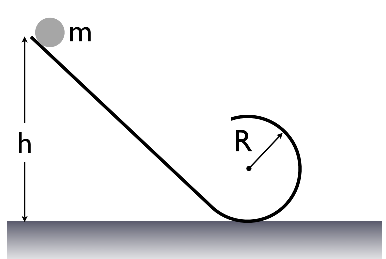
$$\lambda = \frac{v}{f} = \frac{v}{\frac{nv}{2L}} = \frac{2L}{n}$$

Neither L nor n changes, so the new wavelength is the same as the old.

Remember:

wavelength :: spatial periodicity
frequency :: temporal periodicity

8. A solid brass ball of mass m will roll smoothly along a loop-the-loop track when released from rest along the straight section. The circular loop has radius R, and the ball has radius $r \ll R$. What is h if the ball is on the verge of leaving the track when it reaches the top of the loop? Assume the ball has a moment of inertia $I = kmr^2$, $k \in \mathbb{R} \,|\, 0 < k < 1$.



There are two key parts to this problem: conservation of energy fill find the velocity at the top of the loop, and the centripetal acceleration constraint will tell us the minimum value required to stay on the loop. First, we apply conservation of energy.

At the beginning of the track, we have only potential energy. Using the ground level as our zero for gravitational potential energy, we have $E_i = mgh$. At the top of the loop, we have gravitational potential energy, translational kinetic energy, and rotational kinetic energy. Adding all terms,

$$E_f = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mg\left(2R\right)$$

If we have smooth rolling without slipping, then $v = r\omega$. (We must be careful not to confuse the radius of the loop R with the radius of the ball r at this point.) We also know $I = kmr^2$. Finally, conservation of energy dictates $E_i = E_f$. Thus,

$$\begin{aligned}
E_i &= E_f \\
mgh &= \frac{1}{2}mv^2 + \frac{1}{2}kmr^2\left(\frac{v}{r}\right)^2 + mg\left(2R\right) \\
mg\left(h - 2R\right) &= \frac{1}{2}mv^2\left(1 + k\right) \\
h &= \frac{\left(1 + k\right)v^2}{2g} + 2R
\end{aligned}$$

This gives us an equation for the height in terms of the magnitude of the velocity at the top of the loop, but we do not know yet if the velocity is sufficient to keep the ball on the loop in the first place. This will only happen if the sum of all forces acting on the ball is sufficient to provide the necessary centripetal acceleration. Remember, centripetal acceleration is a constraint on all other forces when we consider circular motion – to stay on a circular path of radius R, we require a radial acceleration v^2/R.

In the present case, we have only one force, namely, gravity. At the top of the loop it acts purely in the radial direction, so our constraint equation is

$$\sum F_r = -mg = -\frac{mv^2}{R} \quad \Longrightarrow \quad v^2 = Rg$$

Combining this with our previous expression,

$$h = \frac{\left(1 + k\right)v^2}{2g} + 2R = \frac{1}{2}R\left(1 + k\right) + 2R = \frac{1}{2}\left(5 + k\right)R$$

Note that the radius of the ball does not matter in the end anyway ...