Name & ID ____

University of Alabama
Department of Physics and Astronomy

PH 125 / LeClair Spring 2014

Exam I Solutions

1. A stone is dropped into a river from a bridge 43.9 m above the water. Another stone is thrown vertically down 1.00 s after the first is dropped. The stones strike the water at the same time. What is the initial speed of the second stone? Neglect air resistance.

Solution: Let $t_o = 1\,\mathrm{s}$ be the delay between the first stone dropping and the second stone being thrown, and $h = 43.9\,\mathrm{m}$ be the height both stones start from. We'll call y the vertical axis, with $+y$ being upward. The first stone's coordinate as a function of time is then

$$y_1(t) = h - \frac{1}{2}gt^2 \tag{1}$$

The first stone hits the river when $y_1(t) = 0$, or

$$0 + h - \frac{1}{2}gt^2 \tag{2}$$

$$t_{\text{hit}} = \sqrt{\frac{2h}{g}} \tag{3}$$

This is how long the first stone takes to fall. The second stone takes a time t_o less. The second stone's coordinate as a function of time *from the time it was thrown* downward with initial velocity v_{iy} is

$$y_2(t) = h - v_{iy}t - \frac{1}{2}gt^2 \tag{4}$$

We know the second stone hits at a time $t_{\text{hit}} - t_o$, and thus $y_2(t_{\text{hit}} - t_o) = 0$.

$$y_2(t_{\text{hit}} - t_o) = h - v_{iy}(t_{\text{hit}} - t_o) - \frac{1}{2}g(t_{\text{hit}} - t_o)^2 \tag{5}$$

$$v_{iy}(t_{\text{hit}} - t_o) = h - \frac{1}{2}g(t_{\text{hit}} - t_o)^2 \tag{6}$$

$$v_{iy} = \frac{h - \frac{1}{2}g(t_{\text{hit}} - t_o)^2}{t_{\text{hit}} - t_o} \tag{7}$$

In terms of the original given quantities,

$$v_{iy} = \frac{h - \frac{1}{2}g\left(\sqrt{\frac{2h}{g}} - t_o\right)^2}{\sqrt{\frac{2h}{g}} - t_o} \approx 12.3\,\mathrm{m/s} \tag{8}$$

2. A football kicker can give the ball an initial speed of 25 m/s. What are the **(a)** least and **(b)** greatest elevation angles which he can kick the ball to score a field goal from a point 50.3 m (55 yd) in front of the goalposts whose horizontal bar is 3.35 m (10 ft) above the ground? Neglect air resistance.

Solution: Let the origin $(0,0)$ be the position from which the ball was kicked, with $+y$ vertically upward and $+x$ horizontally in the direction the ball is traveling. We know it must travel a distance $x_m = 50.3\,\text{m}$ and at that point have a height of at least $y_m = 3.35\,\text{m}$. Given that we can specify the ball's entire trajectory $y(x)$, this gives us a constraint: the ball must pass through (x_m, y_m) at least. The trajectory is

$$y(x) = x\tan\theta - \frac{gx^2}{2v_i^2\cos^2\theta} \tag{9}$$

We know the initial speed v_i, and we know the trajectory passes through (x_m, y_m). Using this, we should solve for θ. We should find two values that satisfy the minimum condition, and all angles in between should work.

$$\begin{aligned}
y_m &= x_m\tan\theta - \frac{gx_m^2}{2v_i^2\cos^2\theta} && (10)\\
y_m &= x_m\tan\theta - \frac{gx_m^2\sec^2\theta}{2v_i^2} \qquad \text{note} \quad 1+\tan^2\theta = \sec^2\theta && (11)\\
y_m &= x_m\tan\theta - \frac{gx_m^2}{2v_i^2}\left(1+\tan^2\theta\right) && (12)\\
0 &= -\frac{gx_m^2}{2v_i^2}\tan^2\theta + x_m\tan\theta - \left(y_m + \frac{gx_m^2}{2v_i^2}\right) && (13)
\end{aligned}$$

This is a quadratic equation in $\tan\theta$, which has solution

$$\tan\theta = \frac{-x_m \pm \sqrt{x_m^2 - \frac{2gx_m^2}{v_i^2}\left(y_m + \frac{gx_m^2}{2v_i^2}\right)}}{-gx_m^2/v_i^2} = \frac{v_i^2}{gx_m}\left(1 \pm \sqrt{1 - \frac{2g}{v_i^2}\left(y_m + \frac{gx_m^2}{2v_i^2}\right)}\right) \approx \{31°, 63°\} \tag{14}$$

We know that for an angle θ less than 45°, the projectile's range is always greater for larger angles between θ and 45°, and similarly that for an angle θ greater than 45°, all smaller angles between θ and 45° will give a larger range. Therefore, if $\theta = \{31°, 63°\}$ are the extreme cases where the ball just clears the crossbar (or more precisely, intersects its position), all angles in between will give a larger range and will also clear the bar.

3. A football player punts the football so that it will have a "hang time" (time of flight) of 4.5 s and land 46 m (∼50 yd) away. If the ball leaves the players's foot 1.50 m above the ground, what must be the **(a)** magnitude and **(b)** angle (relative to the horizontal) of the ball's initial velocity? Neglect air resistance.

Solution: We know the ball travels a horizontal distance $x_m = 46\,\text{m}$, and this is only due to its initial velocity along the x axis v_{ix}. Since the ball is in flight for a time $t_o = 4.5\,\text{s}$,

$$\begin{aligned}
x_m &= v_{ix}t_o && (15)\\
v_{ox} &= \frac{x_m}{t_o} && (16)
\end{aligned}$$

We also know that after a time t_o the ball has traveled a net vertical distance $y_m = -1.50\,\text{m}$ downward (from the kicker's foot to the ground). The vertical position of the ball as a function of time will get us v_{iy}.

$$y(t) = v_{iy}t - \frac{1}{2}gt^2 \tag{17}$$

$$y(t_o) = y_m = v_{iy}t_o - \frac{1}{2}gt_o^2 \tag{18}$$

$$v_{iy} = \frac{y_m + \frac{1}{2}gt_o^2}{t_o} \tag{19}$$

The initial speed can then be found from the components of the velocity:

$$v_i = \sqrt{v_{ix}^2 + v_{iy}^2} = \sqrt{\frac{x_m^2}{t_o^2} + \left(\frac{y_m + \frac{1}{2}gt_o^2}{t_o}\right)^2} = \frac{1}{t_o}\sqrt{x_m^2 + \left(y_m + \frac{1}{2}gt^2\right)^2} \approx 24\,\mathrm{m/s} \tag{20}$$

The angle is given by $\tan\theta = v_{iy}/v_{ix}$.

$$\theta = \tan^{-1}\left(\frac{v_{iy}}{v_{ix}}\right) = \tan^{-1}\left(\frac{y_m + \frac{1}{2}gt^2}{x_m}\right) \approx 65^\circ \tag{21}$$

We could have also noticed from the beginning that $\frac{1}{2}gt_o^2 \approx 99\,\mathrm{m}$, much greater than $y_m = -1.5\,\mathrm{m}$. That means that in the terms like $y_m + \frac{1}{2}gt_o^2$ we could safely neglect the y_m term since the second term is so much larger. We would only incur an error of roughly the ratio of the two terms, about 1.5% or so (amounting to about 0.7° if you work it out). That would have made the algebra much simpler, and we would probably not require better than 1% accuracy anyway. It also meshes with our intuition that the small distance above the ground shouldn't make much difference when it pales in comparison to the overall distance traveled.

4. In the figure below, three ballot boxes are connected by cords, one of which wraps over a pulley having negligible friction on its axle and negligible mass. The three masses are $m_a = 30.0\,\mathrm{kg}$, $m_b = 40.0\,\mathrm{kg}$, and $m_c = 10.0\,\mathrm{kg}$. When the assembly is released from rest, **(a)** what is the tension in the cord connecting B and C, and **(b)** how far does A move in the first $0.250\,\mathrm{s}$ (assuming it does not reach the pulley)? The table may be assumed to be frictionless.

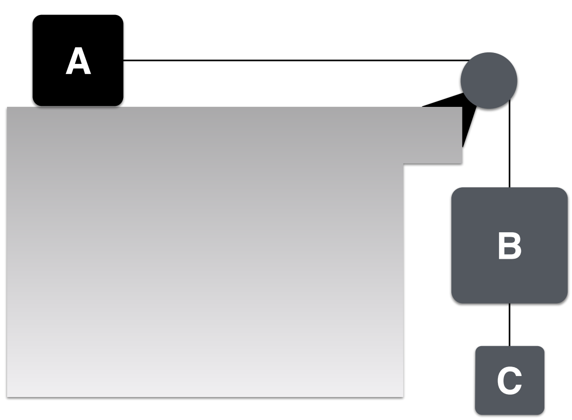


Figure 1: *Three boxes connected by cords, one of which wraps over a pulley.*

Solution: Let the tension in the cord connecting B and C be T_{bc}, and the tension in the cord connecting B and A be T_{ba}. Mass C has only two forces acting on it: T_{bc} and its weight $m_c g$. Clearly the acceleration is downward, in the same direction as the weight and opposite the tension.

$$T_{bc} - m_c g = -m_c a \tag{22}$$

Mass A has only one force acting on it, the tension T_{ab}, giving

$$T_{ab} = m_a a \tag{23}$$

This is not quite enough information. However, since B and C are connected together, we may treat them, from the point of view of the upper cord, as a single mass $(m_b + m_c)$ connected to mass A. There are two forces acting on B and C connected together: their weight, and the tension T_{ab}. Thus,

$$T_{ab} - (m_b + m_c)\, g = -\,(m_b + m_c)\, a \tag{24}$$

Since we already know $T_{ab} = m_a a$,

$$m_a a - (m_b + m_c)\, g = -\,(m_b + m_c)\, a \tag{25}$$

$$a = \left[\frac{m_b + m_c}{m_a + m_b + m_c}\right] g = \frac{5}{8} g \approx 6.13\,\mathrm{m/s^2} \tag{26}$$

The desired tension is readily found now, since $T_{bc} = m_c\,(g - a)$

$$T_{bc} = m_c g - m_c \left[\frac{m_b + m_c}{m_a + m_b + m_c}\right] g = \left[\frac{m_c m_a + m_c m_b + m_c^2 - m_c m_b - m_c^2}{m_a + m_b + m_c}\right] \tag{27}$$

$$T_{bc} = g \left[\frac{m_c m_a}{m_a + m_b + m_c}\right] \approx 36.8\,\mathrm{N} \tag{28}$$

Given an acceleration a, the distance traveled in time t is readily found.

$$\Delta x = \frac{1}{2} a t^2 \approx 0.192\, m \tag{29}$$

5. A puck of mass $m = 1.50\,\mathrm{kg}$ slides in a circle of radius $r = 0.20\,\mathrm{m}$ on a frictionless table while attached to a hanging cylinder of mass $M = 2.50\,\mathrm{kg}$ by means of a cord that extends through a hole in the table (see figure below). What speed keeps the hanging cylinder at rest? *Hint: think about the constraints of the forces acting on m given its path.*

Solution: The mass m on the table has only one force acting on it, the tension T in the cord. Since m moves in a circular path, the sum of all forces must provide the centripetal force mv^2/r. Both it and the tension are pointing toward the center of the circle. Thus, we must have

$$\sum F = T = \frac{mv^2}{r} \tag{30}$$

The hanging mass M has two forces: the tension T pulling it upward, and its weight Mg pulling it downward. The two must balance if the mass is to remain stationary.

$$\sum F = T - Mg = 0 \qquad \Longrightarrow \qquad T = Mg \tag{31}$$

Combining our two results, we must have

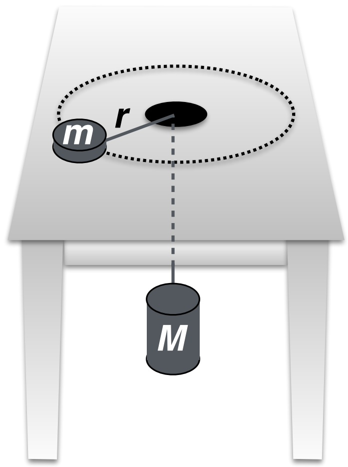


Figure 2: *A puck slides in a circle on a table, holding up a cylindrical mass.*

$$Mg = \frac{mv^2}{r} \tag{32}$$

$$v = \sqrt{\frac{Mgr}{m}} \approx 1.81\,\mathrm{m/s} \tag{33}$$

6. A rifle that shoots a bullet at $460\,\mathrm{m/s}$ is to be aimed at a target $45.7\,\mathrm{m}$ away. If the center of the target is level with the rifle, how high above the target must the rifle barrel be pointed so that the bullet hits dead center?

Solution: **Given:** The magnitude of the initial velocity of a fired bullet v_i and its distance from a target d.

Find: The height above the target that the shooter must aim y_{aim}. We can easily find this once we know the firing angle θ required for the bullet to hit the target. That is, the angle such that the bullet is at the same height a distance d from where it is fired.

Sketch: For convenience, let the origin be at the position the bullet is fired from. Let the $+x$ axis run horizontally, from the bullet to the target, and let the $+y$ axis run vertically. Let time $t = 0$ be the moment the projectile is launched.

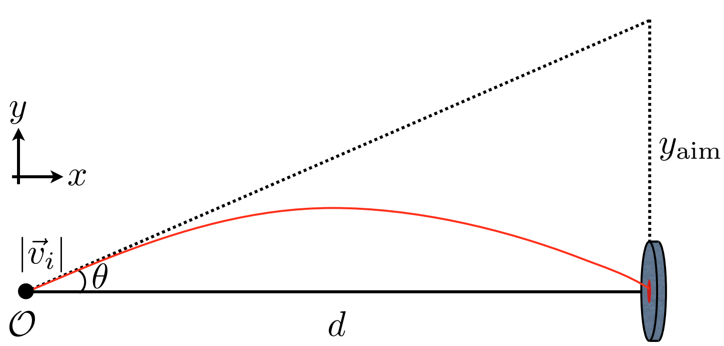


Figure 3: *Firing a rifle at a distant target. The bullet's trajectory is (approximately) shown in red.*

The bullet is fired at an initial velocity $|\vec{\mathbf{v}}_i|$ and angle θ, a distance d from a target. The target is at the same vertical position as the rifle, so we need to find the angle θ and resulting y_{aim} such that the bullet is at $y=0$ at $x=d$.

Relevant equations: In the x direction, we have constant velocity and no acceleration, with position starting at the origin at $t=0$:

$$x(t) = v_{ix}t = |\vec{\mathbf{v}}_i|t\cos\theta \tag{34}$$

In the y direction, we have an initial constant velocity of $v_{iy}=|\vec{\mathbf{v}}_i|\sin\theta$ and a constant acceleration of $a_y=-g$:

$$y(t) = v_{iy}t - \frac{1}{2}gt^2 \tag{35}$$

Solving Eq. 34 for t and substituting into Eq. 35 yields our general projectile equation, giving the path of the projectile $y(x)$ when launched from the origin with initial velocity $|\vec{\mathbf{v}}_i|$ and angle θ above the x axis:

$$y(x) = x\tan\theta - \frac{gx^2}{2|\vec{\mathbf{v}}_i|^2\cos^2\theta} \tag{36}$$

With our chosen coordinate system and origin, $y_o=0$. We also need the aiming height above the target in terms of the target distance and firing angle, which we can get from basic trigonometry:

$$\tan\theta = \frac{y_{\text{aim}}}{d} \tag{37}$$

Note that one can also use the "range equation" directly, but this is less instructive. It is fine for you to do this in your own solutions, but keep in mind you will probably not be given these sort of specialized equations on an exam – you should know how to derive them.

Symbolic solution: We desire the bullet to reach point $(d,0)$. Substituting these coordinates into Eq. 36, and solving for θ:

$$y(x) = x\tan\theta - \frac{gx^2}{2|v_i|^2\cos^2\theta} \tag{38}$$

$$0 = d\tan\theta - \frac{gd^2}{2|v_i|^2\cos^2\theta} \tag{39}$$

$$d\tan\theta = \frac{gd^2}{2|v_i|^2\cos^2\theta} \tag{40}$$

$$\tan\theta\cos^2\theta = \frac{gd}{2|\vec{\mathbf{v}}_i|^2} \tag{41}$$

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta = \frac{gd}{2|\vec{\mathbf{v}}_i|^2} \tag{42}$$

$$\Longrightarrow \theta = \frac{1}{2}\sin^{-1}\left[\frac{gd}{|\vec{\mathbf{v}}_i|^2}\right] \tag{43}$$

Given θ, rearranging Eq. 37 gives us the aiming height:

$$y_{\text{aim}} = d\tan\theta \tag{44}$$

Numeric solution: We are given $|\vec{\mathbf{v}}_i|=460\,\text{m/s}$ and $d=45.7\,\text{m}$:

$$\theta = \frac{1}{2}\sin^{-1}\left[\frac{gd}{|\vec{\mathbf{v}}_i|^2}\right] = \frac{1}{2}\sin^{-1}\left[\frac{\left(9.81\,\mathrm{m/s^2}\right)\left(4.57\,\mathrm{m}\right)}{\left(460\mathrm{m/s}\right)^2}\right] \approx 0.06069^\circ \tag{45}$$

Given the angle, we can find the height above the target we need to aim:

$$y_{\text{aim}} = d\tan\theta \approx (45.7\,\mathrm{m})\tan\left(0.06069^\circ\right) \xrightarrow[\text{digits}]{\text{sign.}} 0.0484\,\mathrm{m} = 4.84\,\mathrm{cm} \tag{46}$$

Double check: One check is use the pre-packaged projectile range equation and make sure that we get the same answer. Given $\theta \approx 0.0607^\circ$, we should calculate a range of d.

$$R = \frac{|\vec{\mathbf{v}}_i|^2 \sin 2\theta}{g} = \frac{(460\,\mathrm{m/s})\,(\sin 0.1214^\circ)}{(9.81\,\mathrm{m/s^2})} \approx 45.7\,\mathrm{m} \tag{47}$$

This is not truly an independent check, since it is derived using the same equations we used above, but it is a nice indication that we haven't gone wrong anywhere.

As a more independent estimate, we can first calculate the time it would take the bullet to reach the target in the absence of gravitational acceleration – if it were just heading straight toward the target at 460 m/s. This is not so far off the real time, since the firing angle is small anyway:

$$t_{\text{est}} = \frac{d}{|\vec{\mathbf{v}}_i|} \approx 0.1\,\mathrm{s} \tag{48}$$

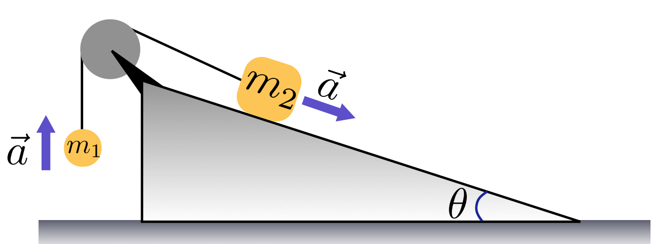
In that time, how far would the bullet fall under the influence of gravity (alone)?

$$y_{\text{fall}} \approx -\frac{1}{2}gt_{\text{est}}^2 \approx 0.05\,\mathrm{m} \tag{49}$$

Thus, we estimate that the bullet should fall about 5 cm on its way to the target, meaning we should aim about 5 cm high, in line with what we calculate by more exact means.

You can also verify that units come out correctly in Eq. 45 and Eq. 46. The argument of the $\sin^{-1}$ function must be dimensionless, as it is, and y_{aim} should come out in meters, as it does. If you carry the units through the entire calculation, or at least solve the problem symbolically, without numbers until the last step, this sort of check is trivial.

7. A ball of mass m_1 and a block of mass m_2 are connected by a lightweight cord that passes over a frictionless pulley of negligible mass, as shown below. The block lies on a frictionless incline of angle θ. Find the magnitude of the acceleration of the two objects. *Note: the direction for $\vec{\mathbf{a}}$ shown is just an arbitrary choice, it implies nothing about the problem.*



Solution: First, make a free body diagram for m_1. We have only weight and tension in the string, and we postulate that acceleration is upward. Let the upward direction be positive. Then it is clear

$$T - m_1 g = m_1 a \qquad \Longrightarrow \qquad T = m_1 (a + g)$$

where T is the tension in the string. For m_2 on the ramp, let the x axis point down the ramp, and the y axis vertically normal to the ramp. Along the ramp direction, a balance of forces reads:

$$\sum F_x = m_2 g \sin\theta - T = m_2 a \qquad \Longrightarrow \qquad a = g\sin\theta - \frac{T}{m_2}$$

Of course, the tension is the same everywhere in the string, and the acceleration is the same for both masses. Solving the first equation for T and substituting into the second, we can solve for the acceleration.

$$\begin{aligned} a &= g\sin\theta - \frac{T}{m_2} = g\sin\theta - \frac{m_1}{m_2}(a+g) \\ a\left(\frac{m_1}{m_2} + 1\right) &= g\sin\theta - \frac{m_1}{m_2} g \\ a &= g\left[\frac{\sin\theta - \frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1}\right] = g\left[\frac{m_2\sin\theta - m_1}{m_1 + m_2}\right] \end{aligned}$$

We can note that so long as $m_2 \sin\theta > m_1$, the acceleration is positive as we have postulated, meaning m_2 slides down the ramp and pulls m_1 upward. If $m_2 \sin\theta < m_1$, the acceleration is negative, and it means the hanging mass m_1 is falling and pulling m_2 up the ramp.

Formula sheet

$$g = 9.81\,\mathrm{m/s^2}$$

$$0 = ax^2 + bx^2 + c \Longrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$1\,\mathrm{N} = 1\,\mathrm{kg}\cdot\mathrm{m/s^2}$$

Vectors:

$$\vec{\mathbf{a}} = a_x\,\hat{\imath} + a_y\,\hat{\jmath} + a_z\,\hat{\mathbf{k}}$$

$$\vec{\mathbf{b}} = b_x\,\hat{\imath} + b_y\,\hat{\jmath} + b_z\,\hat{\mathbf{k}}$$

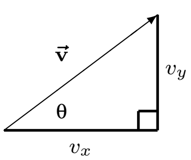
$$|\vec{\mathbf{a}}| = \sqrt{a_x^2 + a_y^2}$$

$$\tan\theta = \frac{a_y}{a_x}$$

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = (a_x + b_x)\,\hat{\imath} + (a_y + b_y)\,\hat{\jmath} + (a_z + b_z)\,\hat{\mathbf{k}}$$

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z$$

$$|\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\varphi$$



$$v_y = |\vec{\mathbf{v}}|\sin\theta$$
$$v_x = |\vec{\mathbf{v}}|\cos\theta$$
$$\tan\theta = \frac{v_y}{v_x}$$

1-D motion:

$$v(t) = \frac{d}{dt}x(t)$$

$$a(t) = \frac{d}{dt}v(t) = \frac{d^2}{dt^2}x(t)$$

const. acc. $\downarrow$

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2$$

$$v_f^2 = v_i^2 + 2a_x\Delta x$$

$$v_f = v_i + at$$

Projectile motion:

$$v_x(t) = v_{ix} = |\vec{\mathbf{v}}_i|\cos\theta$$

$$v_y(t) = |\vec{\mathbf{v}}_i|\sin\theta - gt = v_{iy}\sin\theta - gt$$

$$x(t) = x_i + v_{ix}t$$

$$y(t) = y_i + v_{iy}t - \frac{1}{2}gt^2$$

$$y(x) = x\tan\theta - \frac{gx^2}{2|\vec{\mathbf{v}}_i|^2\cos^2\theta}$$

over level ground:

$$\text{max height } = H = \frac{v_i^2\sin^2\theta_i}{2g}$$

$$\text{Range } = R = \frac{v_i^2\sin 2\theta_i}{g}$$

Force:

$$\sum\vec{\mathbf{F}} = \vec{\mathbf{F}}_{\text{net}} = m\vec{\mathbf{a}}$$

$$\sum F_x = ma_x$$

$$\sum F_y = ma_y$$

$$F_{\text{grav}} = mg = \text{weight}$$

$$\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$$

$$f_s \le \mu_s n$$

$$f_{s,\max} = \mu_s n$$

$$f_k = \mu_k n$$

$$\vec{\mathbf{F}}_{\text{centr.}} = -\frac{mv^2}{r}\,\hat{\mathbf{r}} \quad \text{circular}$$

2-D motion:

$$\vec{\mathbf{r}} = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$$

$$x(t) = x_i + v_{ix}t + \frac{1}{2}a_x t^2$$

$$y(t) = y_i + v_{iy}t + \frac{1}{2}a_y t^2$$

$$\vec{\mathbf{v}} = v_x(t)\,\hat{\imath} + v_y(t)\,\hat{\jmath}$$

$$v_x(t) = \frac{dx}{dt} = v_{xi} + a_x t$$

$$v_y(t) = \frac{dy}{dt} = v_{yi} + a_y t$$

$$\vec{\mathbf{a}} = a_x(t)\,\hat{\imath} + a_y(t)\,\hat{\jmath}$$

$$a_x(t) = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$$

$$\vec{\mathbf{a}}_c = -\frac{v^2}{r}\,\hat{\mathbf{r}} \quad \text{circ.}$$

$$T = \frac{2\pi r}{v} \quad \text{circ.}$$

Math:

$$ax^2 + bx^2 + c = 0 \Longrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha \pm \beta)\cos\frac{1}{2}(\alpha \mp \beta)$$

$$\cos\alpha \pm \cos\beta = 2\cos\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta)$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta_{ab}$$

$$\frac{d}{dx}\sin ax = a\cos ax \qquad \frac{d}{dx}\cos ax = -a\sin ax$$

Power	Prefix	Abbreviation
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	μ
10^{-3}	milli	m
10^{-2}	centi	c
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T