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Exam II Solution

1. For an isolated system, conservation of energy holds, and the total energy of the system may be written as the sum of kinetic and potential energies, $E_{\text{tot}} = K + U$. Presume the potential energy U is a function of position alone, $U(x)$. Show that to move between positions x_i and x_f given $U(x)$ the time required is

$$t = \sqrt{\frac{m}{2}} \int\limits_{x_i}^{x_f} \frac{dx}{\sqrt{E_{\text{tot}} - U(x)}} \tag{1}$$

To begin, note that $K = E_{\text{tot}} - U(x) = \frac{1}{2}mv^2$ and $v = \frac{dx}{dt}$.

Solution: Start by solving the conservation of energy equation for kinetic energy, and subsequently dx/dt.

$$K = E - U(x) = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \tag{2}$$

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}\left(E - U(x)\right)} \tag{3}$$

So long as $x(t)$ is a function of a single variable, we can manipulate the differentials like fractions and separate the equation.[i] Proceeding that way would give

$$dt = \frac{dx}{\sqrt{\frac{2}{m}\left(E - U(x)\right)}} \tag{4}$$

$$\int\limits_0^t dt = \int\limits_{x_i}^{x_f} \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{(E - U(x))}} \tag{5}$$

$$t = \sqrt{\frac{m}{2}} \int\limits_{x_i}^{x_f} \frac{dx}{\sqrt{(E - U(x))}} \tag{6}$$

Of course, we really aren't supposed to manipulate differentials like fractions. Better would be to invert and then integrate both sides of Eq. 3 with respect to x.

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}\left(E - U(x)\right)} \tag{7}$$

$$\frac{dt}{dx} = \sqrt{\frac{m}{2}} \frac{1}{\sqrt{E - U(x)}} \tag{8}$$

$$\int\limits_{x_i}^{x_f} \frac{dt}{dx}\,dx = \int\limits_{x_i}^{x_f} \sqrt{\frac{m}{2}} \frac{1}{\sqrt{E - U(x)}} \tag{9}$$

$$t_f - t_i \equiv t = \int\limits_{x_i}^{x_f} \sqrt{\frac{m}{2}} \frac{1}{\sqrt{E - U(x)}} \tag{10}$$

[i] See `http://www.theshapeofmath.com/oxford/physics/year1/calc/sepdiff` for why this is in general *not* a good idea.

For the last line, we basically said $t_i\!=\!0$ and $t_f\!=\!t$. The result is the same as the other approach, but this way we don't rely on the somewhat sketchy process of manipulating differentials (though it works perfectly fine in 1D). All we relied on is that $dx/dt\!=\!1/(dt/dx)$.

2. To stretch a spring a distance d from equilibrium takes an amount W_o of work. **(a)** How much work does it take to stretch the spring from d to $2d$ from equilibrium? **(b)** From Nd to $(N+1)d$ from equilibrium?

Solution: The work required is the change in the spring's potential energy. Given that for a spring $U(x)\!=\!\frac{1}{2}kx^2$, going from equilibrium ($x\!=\!0$) to $x\!=\!d$ requires

$$W_{0\to d} = U(d) - U(0) = \frac{1}{2}kd^2 - 0 = \frac{1}{2}kd^2 \equiv W_o \tag{11}$$

Going from d to $2d$, we still just find the change in potential energy.

$$W_{d\to 2d} = U(2d) - U(d) = \frac{1}{2}k\left(2d\right)^2 - \frac{1}{2}kd^2 = 3\cdot\frac{1}{2}kd^2 = 3W_o \tag{12}$$

Going from Nd to $(N+1)d$,

$$W_{Nd\to(N+1)d} = U\left([N+1]d\right) - U(Nd) = \frac{1}{2}k\left(\left(N+1\right)d\right)^2 - \frac{1}{2}k\left(Nd\right)^2 \tag{13}$$

$$= \frac{1}{2}k\left(N^2+2N+1\right)d^2 - \frac{1}{2}kN^2d^2 = \left(2N+1\right)\cdot\frac{1}{2}kd^2 = \left(2N+1\right)W_o \tag{14}$$

3. A stone is tied to a string of length R. A person whirls this stone in a vertical circle. Assume that the energy of the stone remains constant as it moves around the circle. Show that if the string is to remain taut at the top of the circle, the speed of the stone at the bottom of the circle must be at least $\sqrt{5gR}$.

Solution: At the top of the motion, a force balance would have the tension T and weight mg both pointing downward toward the center of the circle, and the combination of these two forces must provide the centripetal force (which also points toward the center of the circle) to have circular motion. If the speed at the top of the motion is v_t,

$$\sum F = -T - mg = -\frac{mv_t^2}{R} \tag{15}$$

The string will remain taut so long as $T\!>\!0$, so setting $T\!=\!0$ gives us the minimum velocity.

$$\sum F = 0 - mg = -\frac{mv_t^2}{R} \tag{16}$$

$$v_t = \sqrt{gR} \tag{17}$$

Conservation of energy gives us the speed at the bottom. Let the bottom of the motion be $U_{\text{grav}}\!=\!0$ and the speed at the bottom be v_b.

$$
\begin{aligned}
E_{\text{bottom}} &= E_{\text{top}} && (18)\\
\frac{1}{2}mv_b^2 &= \frac{1}{2}mv_t^2 + mg(2R) && (19)\\
v_b^2 &= v_t^2 + 4gR = gR + 4gR = 5gR && (20)\\
v_b &= \sqrt{5gR} && (21)
\end{aligned}
$$

4. A 2.5 g Ping-Pong ball is dropped from a window and strikes the ground 20 m below with a speed of 9.0 m/s. What fraction of its initial potential energy was lost to air friction?

Solution: The ball starts with gravitational potential energy mgh if h is the height above ground and we let $U=0$ at the ground. We expect without air resistance that the ball would have a kinetic energy once it reaches the ground of $\frac{1}{2}mv^2=mgh$ since conservation of energy would apply without air resistance. In actuality, given a speed v_{act} with air resistance acting, we have a kinetic energy of $\frac{1}{2}mv_{\text{act}}^2$. The difference between this and the starting energy (which is the same as the expected kinetic energy) is what we're looking for.

$$
\text{percent lost} = \frac{E_i - E_f}{E_f} = \frac{mgh - \frac{1}{2}mv_{\text{act}}^2}{mgh} = 1 - \frac{v_{\text{act}}^2}{2gh} \approx 79\% \tag{22}
$$

5. A package is dropped onto a horizontal conveyor belt. The mass of the package is m, the speed of the conveyor belt is v, and the coefficient of kinetic friction for the package on the belt is μ_k. **(a)** For what length of time will the package slide on the belt? **(b)** How far does it move in this time? **(c)** How much energy is dissipated by friction? *Hint: would it make any difference if the belt were stationary and the box were moving at velocity v?*

Solution: The key to this problem is to realize that it doesn't matter whether the conveyor belt moves or the box moves, only the relative velocity between them makes a difference. That is, once the box hits the belt it would be precisely the same physics to say that the box is moving with velocity v along it, and we wish to know when the box comes to a stop. Given the normal force on the box is mg, the frictional force resisting the box's motion is $f_k=\mu_k mg=ma$, so there is an acceleration of $\mu_K g$ in the direction opposite the motion. If the box starts with velocity v relative to the conveyor belt,

$$
\begin{aligned}
v_f &= 0 = v_i + at && (23)\\
0 &= v - \mu_k g t && (24)\\
t &= \frac{v}{\mu_k g} && (25)
\end{aligned}
$$

The distance covered we could get from conservation of energy (accounting for the work done by friction $W_f=f_k\Delta x$ of course), or our kinematic equations.

$$\text{kinematics} \quad v_f^2 = v_i^2 + 2a\Delta x = 0 \tag{26}$$

$$\Delta x = \frac{v_i^2}{2a} = \frac{v^2}{2\mu_k g} \tag{27}$$

$$\text{work-energy} \quad \frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2 + W_f = \frac{1}{2}mv_f^2 + \mu_k mg\Delta x \tag{28}$$

$$\Delta x = \frac{v_i^2}{2a} = \frac{v^2}{2\mu_k g} \tag{29}$$

$$\tag{30}$$

How much energy is dissipated by friction? The amount of energy it takes to change the box's velocity from 0 to v, taking the viewpoint that the conveyor is moving, or from v to 0 taking the viewpoint that the box is moving and the conveyor stationary. Either way,

$$\Delta E = W_f = \Delta K = \frac{1}{2}mv^2 \tag{31}$$

6. An alternative to using Newton's laws and forces to analyze mechanical problems is to use an energy-based approach known as the Lagrangian method. The Lagrangian function of a system is defined as the *difference* between kinetic and potential energy:

$$L = K - U = \frac{1}{2}mv^2 - U \tag{32}$$

Here v is the particle's velocity and U its (in general position-dependent) potential energy. The path a particle takes is the one that extremizes this Lagrangian function, integrated over the entire path. That condition can be summarized by the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} = 0 \tag{33}$$

(The ∂ symbol means 'take the derivative with respect to the given variable leaving all other variables constant.' If this is unfamiliar, just treat it like a normal derivative.)

Show that the above two equations reproduce Newton's 2nd law ($\sum F = ma$) for **(a)** a particle under the influence of gravity alone, $U(x) = mgx$, and **(b)** a particle interacting with a spring, $U(x) = \frac{1}{2}kx^2$. **(c)** Now consider an arbitrary $U(x)$. What relationship between U and the net force can be derived from the above equations? Note: writing $U(x)$ means U is a function of x **only**.

Solution: This one is just math, but shows you a bit of the neatness of the Lagrangian approach. Rather than fiddle with forces and stuff like that, what you're essentially doing is an optimization process to find the particle's path, and it requires only taking derivatives. You can find some nice notes at

`http://www.people.fas.harvard.edu/~djmorin/chap6.pdf`

Keep in mind they use the notation that a dot over a variable represents differentiation with respect to time, i.e., $\frac{dx}{dt} = \dot{x}$ and $\frac{d^2x}{dt^2} = \ddot{x}$.

For a gravitational potential energy, $U(x) = mgx$. Thus,

$$L = \frac{1}{2}mv^2 - mgx \tag{34}$$
$$\frac{\partial L}{\partial v} = mv \tag{35}$$
$$\frac{d}{dt}\frac{\partial L}{\partial v} = \frac{d}{dt}(mv) = ma \quad \text{(presume } m \text{ is constant)} \tag{36}$$
$$\frac{\partial L}{\partial x} = -mg \tag{37}$$
$$0 = \frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} = ma + mg \tag{38}$$
$$ma = -mg \tag{39}$$

This approach yields the familiar result from Newton's second law, the gravitational force is $F = ma = -mg$. For the spring, $L = \frac{1}{2}mv^2 - \frac{1}{2}kx^2$.

$$L = \frac{1}{2}mv^2 - \frac{1}{2}kx^2 \tag{40}$$
$$\frac{\partial L}{\partial v} = mv \tag{41}$$
$$\frac{d}{dt}\frac{\partial L}{\partial v} = ma \tag{42}$$
$$\frac{\partial L}{\partial x} = -kx \tag{43}$$
$$0 = \frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} = ma + kx \tag{44}$$
$$ma = -kx \tag{45}$$

Again, we recover the same result we would have with Newton's law, the spring force is $-kx$. For a generic potential which depends only on x, $L = \frac{1}{2}kx^2 - U(x)$.

$$L = \frac{1}{2}mv^2 - mgx \tag{46}$$
$$\frac{\partial L}{\partial v} = mv \tag{47}$$
$$\frac{d}{dt}\frac{\partial L}{\partial v} = ma \tag{48}$$
$$\frac{\partial L}{\partial x} = -\frac{\partial U(x)}{\partial x} \tag{49}$$
$$0 = \frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} = ma + \frac{\partial U(x)}{\partial x} \tag{50}$$
$$ma = -\frac{\partial U(x)}{\partial x} \tag{51}$$

Here we recover the general 1D result that says so long as U depends only on position (making it a conservative force in 1D at least), the force can be found from the potential energy, $F = ma = -\frac{\partial U(x)}{\partial x}$.