Name & ID

PH125 Exam IV: Solutions

1. The space shuttle releases a 470 kg satellite while in an orbit 280 km above the surface of the earth. A rocket engine on the satellite boosts it to a geosynchronous orbit. How much energy is required for the orbit boost? (Note: the earth's radius is 6378 km, its mass is 5.98×10^{24} kg, and $G = 6.67 \times 10^{-11} N \cdot m^2 kg^{-2}$. Hint: "geosynchronous" means the satellite's period T is 24 hrs.)

Solution: For a geosynchronous orbit, the period T is 24 hr. Using Kepler's law, we can find the distance from the earth's center for this orbit:

$$T^2 = \frac{4\pi^2 rg^3}{GM_e} \tag{1}$$

$$r_g = \sqrt[3]{\frac{GMT^2}{4\pi^2}} \approx 42,300\,\text{km} \tag{2}$$

Thus, the satellite changes its orbit to $42,300$ km starting from $h = 280$ km above the earth's surface, a distance $R_e + h$ from the earth's center. Remember that it is the distance from the earth's center, not its surface, which is important for gravitation. This will clearly change the satellite's potential energy, but its kinetic energy will also change. Fortunately, we know the total energy (kinetic plus potential) of an orbiting body of mass m is $E_{\text{tot}} = -\frac{1}{2}\frac{GM_e m}{r}$. The change in energy is thus

$$\Delta E_{\text{tot}} = -\frac{1}{2}\frac{GM_e m}{r_g} - \left(-\frac{1}{2}\frac{GM_e m}{R_e + h}\right) \approx 1.2 \times 10^{10}\,\text{J} \tag{3}$$

2. Calculate the mass of the Sun given that the Earth's distance from the Sun is 1.496×10^{11}m. (Hint: you already know the period of the Earth's orbit.)

Solution: We know the earth's period of rotation T is about 365 days, or about 3.15×10^7 s. Given the earth's orbital distance r, we can use Kepler's law to find the mass of the sun M_s.

$$T^2 = \frac{4\pi^2 r^3}{GM_s} \tag{4}$$

$$M_s = \frac{4\pi^2 r^3}{GT^2} \approx 2 \times 10^{30}\,\text{kg} \tag{5}$$

3. The free-fall acceleration on the surface of the Moon is about one sixth of that on the surface of the Earth. If the radius of the Moon is about $0.250\,R_E$, find the ratio of their average densities, $\rho_{\text{Moon}}/\rho_{\text{Earth}}$.

Solution: The force on an object of mass m a distance r from the earth's center is

$$F_e = -\frac{GM_e m}{r^2} \approx m g_e \tag{6}$$

Near the earth's surface, this force is approximately constant, and we make the identification $g_e = GM_e/r^2$. On the moon, the same mass m a distance r from the center of the moon would feel a force

$$F_g = -\frac{GM_m m}{r^2} \approx m g_m \tag{7}$$

Thus, $g_m = GM_m/r^2$. We are told $g_m/g_e = 1/6$, and we know the masses in terms of the average densities:

$$M_m = \frac{4}{3}\pi R_m^3 \rho_m \tag{8}$$

$$M_e = \frac{4}{3}\pi R_e^3 \rho_e \tag{9}$$

The ratio g_m/g_e will then allow us to determine the ratio of the densities:

$$\frac{g_m}{g_e} = \frac{\rho_m}{\rho_e}\left(\frac{R_e}{R_m}\right)^2\left(\frac{R_m}{R_e}\right)^3 = \frac{R_m \rho_m}{R_e \rho_e} = \frac{1}{6} \tag{10}$$

Since we are told $R_m = \frac{1}{4}R_3$, this gives

$$\frac{\rho_m}{\rho_e} = \frac{g_m R_e}{g_e R_m} = \frac{1}{6} \cdot \frac{4}{1} = \frac{2}{3} \tag{11}$$

4. In the figure below, two masses are connected to each other and vertical walls by three identical springs. Presume $m_1 = m_2$ for simplicity. As it turns out, there are two stable frequencies of oscillation of the system. Find one of them. *Hint: there are two obvious ways the two masses can move relative to each other. One of them is really simple.*

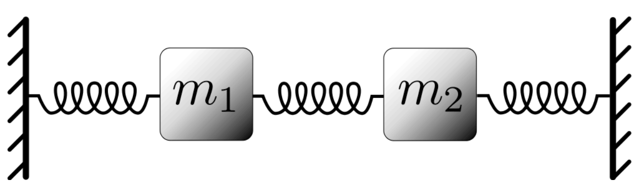

Figure 1: *From* *http://en.wikipedia.org/wiki/Normal_mode.*

Solution: There are only two stable modes of oscillation that can persist: either the masses move in unison, or they move perfectly in opposition to one another. If the masses move in unison, the central spring has no displacement, and there is no net force between the two masses. If that is the case, each mass behaves as though it is only connected by a single spring of constant k, and the frequency is just $\omega = \sqrt{k/m}$. (The other frequency, with the masses moving perfectly out of phase, ends up being $\omega = \sqrt{3k/m}$.)

Recall that we solved this in HW6, see the solution to problem 9, in particular the special case of equal masses and springs on page 12.

5. *Energetics of diatomic systems.* An expression for the potential energy of two neutral atoms as a function of their separation x is given by the *Morse potential*,

$$U(r) = U_o\left[1 - e^{-a(r-r_o)}\right]^2 \tag{12}$$

where x_o is the equilibrium spacing. Calculate the force constant for small oscillations about $r = r_o$. *Hint: At equilibrium, the net force is zero. For small δ, one may approximate $e^{\delta} \approx 1 + \delta + \frac{1}{2}\delta^2 + \cdots$.*

Solution: Just for fun, let's find the equilibrium spacing and dissociation energy too. Equilibrium is in general characterized by a net force of zero, or a minimum of potential energy: $F = -dU/dr = 0$. Thus, we find the equilibrium spacing by figuring out at what radius (or radii) this condition is met.

$$\frac{dU}{dr} = 2U_o\left[1 - e^{-a(r-r_o)}\right]\left(ae^{-a(r-r_o)}\right) = 0 \tag{13}$$

Either of the terms in brackets could be zero. The latter only leads to the trivial solution of $r \to \infty$, meaning there is no molecule in the first place. Setting the former term in brackets to zero,

$$0 = 1 - e^{-a(r-r_o)} \qquad \Longrightarrow \qquad r = r_o \tag{14}$$

The equilibrium spacing is just the parameter r_o in the potential energy function, which is nice. The dissociation energy is defined as the amount of energy required to take the system from equilibrium at $r = r_o$ to complete breakup for $r \to \infty$. Thus,

$$\text{(dissociation energy)} = \left[\lim_{r\to\infty} U(r)\right] - U(r_o) = U_o - 0 = U_o \tag{15}$$

In other words, an amount of work U_o is required to bring about an infinite separation of the atoms, and this defines the dissociation energy.

If we wish to calculate a force constant, it is necessary to show that the force at least approximately obeys Hooke's law for small displacements, i.e., for a small displacement δ from equilibrium, $\delta = r - r_o$, $F(r_o + \delta) \approx k\delta$ where k is the force constant.[i] We have already calculated the force versus displacement:

[i]Equivalently, we could show $U(\delta) \approx \frac{1}{2}k\delta^2$.

$$F(r) = -\frac{dU}{dr} = -2U_o\left[1-e^{-a(r-r_o)}\right]\left(ae^{-a(r-r_o)}\right) = -2U_oa\left(e^{-a(r-r_o)}-e^{-2a(r-r_o)}\right)$$
$$F(r_o+\delta) = -2U_oa\left(e^{-a\delta}-e^{-2a\delta}\right) \tag{16}$$

For small δ, we may make use of the approximation $e^{\delta}\approx 1+\delta+\frac{1}{2}\delta^2+\cdots$. Retaining terms only up to first order,

$$F(r_o+\delta) \approx -2U_oa\left(1-a\delta-1+2a\delta\right) = -\left(2U_oa^2\right)\delta \qquad \Longrightarrow \qquad k=2U_oa^2 \tag{17}$$

Thus, for small displacements from equilibrium, we may treat the molecule as a mass-spring system, with an effective force constant k. Note that we could have equivalently used the method from the last problem, $k=U''$, but it is worth seeing how to approach the problem in a different way. For further information, the Wikipedia article is quite informative:

`http://en.wikipedia.org/wiki/Morse_potential`

Numbers & units:

$g = 9.81\,\mathrm{m/s^2}$ $\quad M_e = 5.96\times10^{24}\,\mathrm{kg}$ $\quad\leftarrow$ earth

$R_e = 6.37\times10^{6}\,\mathrm{m}$ $\quad\leftarrow$ earth $\quad G = 6.67\times10^{11}\,\mathrm{N\,m^2/kg^2}$

Math:

$$ax^2 + bx^2 + c = 0 \Longrightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha\pm\beta)\cos\frac{1}{2}(\alpha\mp\beta)$$

$$\cos\alpha \pm \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta_{ab}$$

$$\frac{d}{dx}\sin ax = a\cos ax \qquad \frac{d}{dx}\cos ax = -a\sin ax$$

$$\int \cos ax\,\mathrm{dx} = \frac{1}{a}\sin ax \qquad \int \sin ax\,\mathrm{dx} = -\frac{1}{a}\cos ax$$

$$\sin\theta \approx \theta \qquad \text{small } \theta$$

$$\cos\theta \approx 1 - \frac{1}{2}\theta^2$$

$$\vec{\mathbf{a}}(t) = \frac{d^2s}{dt^2}\,\hat{\mathbf{T}} + \kappa|\vec{\mathbf{v}}|^2\,\hat{\mathbf{N}} = \frac{d^2s}{dt^2}\,\hat{\mathbf{T}} + \frac{|\vec{\mathbf{v}}|^2}{R}\,\hat{\mathbf{N}} \equiv a_N\,\hat{\mathbf{T}} + a_T\,\hat{\mathbf{N}}$$

Vectors:

$$|\vec{\mathbf{F}}| = \sqrt{F_x^2 + F_y^2} \quad \text{magnitude}$$

$$\theta = \tan^{-1}\left[\frac{F_y}{F_x}\right] \quad \text{direction}$$

$$d\vec{\mathbf{l}} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$$

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = a_xb_x + a_yb_y + a_zb_z = \sum_{i=1}^{n} a_ib_i = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta$$

$$\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad |\vec{\mathbf{a}}\times\vec{\mathbf{b}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin\theta$$

Waves:

$$y_{\rightarrow}(x,t) = y_m\sin(kx-\omega t) \qquad k = \frac{2\pi}{\lambda}$$

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f \quad \text{wave speed}$$

$$v = \sqrt{T/\mu} \quad \mu = M/L$$

$$P_{avg} = \frac{1}{2}\mu v\omega^2 y_m^2 \quad \text{pwr}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad \text{wave}$$

$$y_t = 2y_m\cos\frac{\varphi}{2}\sin(kx-\omega t+\varphi/2) \quad \text{int.}$$

$$y_t = 2y_m\sin kx\cos\omega t \quad \text{standing}$$

$$f = \frac{v}{\lambda} = \frac{nv}{2L} \quad n\in\mathbb{N}$$

Rotation: we use radians

$$s = \theta r \quad \leftarrow \text{arclength}$$

$$\omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d\omega}{dt}$$

$$a_t = \alpha r \quad \text{tangential} \qquad a_r = \frac{v^2}{r} = \omega^2 r \quad \text{radial}$$

$$I = \sum_i m_i r_i^2 \Rightarrow \int r^2\,dm = kmr^2$$

$$I_z = I_{com} + md^2 \quad \text{axis } z \text{ parallel, dist } d$$

$$\tau_{net} = \sum \vec{\boldsymbol{\tau}} = I\vec{\boldsymbol{\alpha}} = \frac{d\vec{\mathbf{L}}}{dt}$$

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}}\times\vec{\mathbf{F}} \qquad |\vec{\boldsymbol{\tau}}| = rF\sin\theta_{rF}$$

$$\vec{\mathbf{L}} = \vec{\mathbf{r}}\times\vec{\mathbf{p}} = I\vec{\boldsymbol{\omega}}$$

$$K = \frac{1}{2}I\omega^2 = L^2/2I$$

$$\Delta K = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W = \int \tau\,d\theta$$

$$P = \frac{dW}{dt} = \tau\omega$$

Gravitation:

$$\vec{\mathbf{F}}_{12} = \frac{Gm_1m_2}{r^2}\,\hat{\mathbf{r}}_{12} = -\vec{\boldsymbol{\nabla}}U_g$$

$$g = \frac{GM_e}{R_e^2}$$

$$U_g(r) = -\int F(r)\,dr = \frac{-GMm}{r}$$

$$K + U_g = 0 \quad \text{escape} \qquad K + U_g < 0 \quad \text{bound}$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2m} \qquad T^2 = \left(\frac{4\pi^2}{GM}\right)r^3$$

$$E_{\text{orbit}} = \frac{-GMm}{2a} \quad \text{elliptical; } a\to r \text{ for circular}$$

Oscillations:

$$T = \frac{1}{f} \quad \omega = \frac{2\pi}{T} = 2\pi f$$

$$x(t) = x_m\cos(\omega t + \varphi)$$

$$a = -\omega^2 x \qquad \frac{d^2q}{dt^2} = -\omega^2 q$$

$$\omega = \sqrt{k/m} \quad \text{linear osc.}$$

$$T = \begin{cases} 2\pi\sqrt{I/\kappa} & \text{torsion pendulum} \\ 2\pi\sqrt{L/g} & \text{simple pendulum} \\ 2\pi\sqrt{I/mgh} & \text{physical pendulum} \end{cases}$$

$$U = -\frac{1}{2}kx^2 \quad U = -\frac{1}{2}\kappa\theta^2 \quad F = -\frac{dU}{dx} = ma \quad \text{SHM}$$

$$x(t) = x_m e^{-bt/2m}\cos\left(\omega' t + \varphi\right) \quad \text{damped}$$

$$\omega' = \sqrt{k/m - b^2/4m}$$