UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

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Problem set 10, number 9

How do you get the time it takes the object to fall? If you did part (b) correctly, you used conservation of energy to find the velocity of the object as a function of its height above the planet's surface, $v(y)$, which involves a nasty function like

$$v(y) = (\text{constant})\sqrt{\frac{y}{R+y}}$$

If you know the velocity as a function of distance, it is "straightforward" to get the time:

$$\begin{aligned} v(y) &= \frac{dy}{dt} \\ dt &= \frac{dy}{v(y)} \\ \delta t &= \int \frac{dy}{v(y)} = (\text{constant}) \int \sqrt{\frac{R+y}{y}}\, dy \end{aligned}$$

The limits of integration are from $y = h + R$, the height above the planet's surface, to $y = R$, the planet's surface. What you are after, then, is an integral of this generic form:[i]

$$\int \sqrt{\frac{a^2+y}{y}}\, dy$$

Make no mistake: this looks simple enough, but it is *nasty.* If you are game for an analytic solution, try a substitution like $u^2 = y$, and integrate by parts. This gives you one closed function (involving square roots) plus another nasty integral. Follow this by a substitution $u = a\tan\theta$ in the remaining integral. This will give you an integral that looks like

$$\int \sec^3\theta$$

This is a classic Cal-II (or Cal-III) example of an integral that looks simple enough, but requires clever tricks involving more integration by parts. It is subtle to evaluate, enough so that it merits its own Wikipedia article: `http://en.wikipedia.org/wiki/Integral_of_secant_cubed`. On the other hand, it is probably one your calculus teacher sprung on you as an example, just because its that cool when it works out.

If you got that far, congrats! You will have to do the integral above twice ... but if you do it well, and work back through both substitutions, your result should be something like this:

$$\int \sqrt{\frac{a^2+y}{y}}\, dy = \sqrt{y\left(a^2+y\right)} + a^2 \ln\left[\sqrt{a^2+y} + \sqrt{y}\right] + \text{const.}$$

[i]We are using a^2 as our constant instead of a to make subsequent trigonometric substitutions a bit more obvious. It is only an aesthetic choice, and since $R > 0$ it does not cause any trouble. You can leave it as $\sqrt{(R+y)/y}$ or $\sqrt{(a+y)/y}$ and it works out the same way.

(If you end up with stray factors of $1/a$ in the integral, use your knowledge of logarithms to push them away into the "const." term.) Evaluate this at the appropriate limits, and you should have δt.

If you analytically solve this nasty thing from start to finish, clearly elucidating all of your steps, there will be **double points** given for this problem (i.e., bonus points). It is also fair to use `http://integrals.wolfram.com` to get the analytic solution, though the Wolfram Integrator does not simplify the result as much as it ought to.

If you find yourself with extra time ... additional double points if you can show that this reduces to our earlier result in the limit that $h \ll R$, viz., $\frac{1}{2}gt^2 = h$. It is also not easy.