UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 125 / LeClair Spring 2009

Problem Set 10 Solutions

1. Halliday, Resnick & Walker Problem 10.54

We can find the angular acceleration by finding the net torque and using the known moment of inertia:

$$\sum \vec{\boldsymbol{\tau}} = I\vec{\boldsymbol{\alpha}} \quad \text{or} \quad \sum |\vec{\boldsymbol{\tau}}| = I|\vec{\boldsymbol{\alpha}}|$$

In order to find net torque, we need to sum the individual torques resulting from each of the four forces present. Let the positive z axis ($\hat{\mathbf{z}}$ direction) be along the rotation axis out of the plane of the page (or down and to the left, depending on your perspective), and define counter-clockwise rotations to be positive.

Forces $\vec{\mathbf{F}}_2$ and $\vec{\mathbf{F}}_3$ will try to cause clockwise rotations. This means the corresponding torques are along $-\hat{\mathbf{z}}$, or that the magnitude of their torques are negative. Force $\vec{\mathbf{F}}_1$ will try to cause a counter-clockwise rotation, meaning its corresponding torque is along $\hat{\mathbf{z}}$ and its magnitude is positive. Force $\vec{\mathbf{F}}_4$ acts directly on the center of mass, and will result in no torque.

Each individual torque is the vector product of force and the displacement between its application and the axis of rotation. If we draw a vector $\vec{\mathbf{r}}_i$ from the axis of rotation to the point of application for force $\vec{\mathbf{F}}_i$, the magnitude of the vector product between $\vec{\mathbf{F}}_i$ and $\vec{\mathbf{r}}_i$ is simply $F_i r_i \sin\theta_i$, where θ_i is the angle between $\vec{\mathbf{F}}_i$ and $\vec{\mathbf{r}}_i$. Given our convention for positive rotation, force 1 this angle is 90°, while for forces 2 and 3 the angle is 270°. Force 4 has $\theta_4 = 0$. Adding the magnitudes of the individual torques,

$$\sum |\vec{\boldsymbol{\tau}}| = \sum |\vec{\mathbf{r}}_i \times \vec{\mathbf{F}}_i| = F_1 R - F_2 R - F_3 R + F_4(0) = (F_1 - F_2)\, R - F_3 r$$

The moment of inertia for a solid disk is $\frac{1}{2}MR^2$. Using our first equation,

$$\alpha = \frac{\sum \tau}{I} = \frac{(F_1 - F_2)\, R - F_3 r}{\frac{1}{2}MR^2} = \frac{2\,(F_1 - F_2)\, R - F_3 r}{MR^2} \approx 9.72\,\mathrm{rad/s^2}$$

The angular acceleration is positive, consistent with a **counter-clockwise** rotation.

2. Halliday, Resnick & Walker Problem 10.67

Though it may not seem obvious at first, an energy-based approach is somewhat easier in this case. First things first. When the chimney falls, any point along its length a distance r from the base will describe circular motion with radius r. Therefore, all we need to consider is circular motion, albeit with a varying angular acceleration.

We already know the radial (normal; a_r) and tangential (a_t) components of acceleration required for circular motion in terms of the linear velocity v, angular velocity $\omega = d\theta/dt =\! v/r$, angular acceleration $\alpha = d^2\theta/dt^2$, and distance from the circle's center:

$$a_t = \frac{d^2 s}{dt^2} = r\alpha$$
$$a_r = \frac{v^2}{r} = r\omega^2$$

Here s is the length of the path covered by a particular point throughout the motion. For circular motion at radius r through an angle θ, this is just the arc length $s = \theta r$. We will need both α and ω to find a_t and a_r. We can find ω from conservation of energy, and differentiate it with respect to time to find α.

Let the chimney have length l, and define the $\hat{\mathbf{y}}$ direction to be vertical with the origin at the base of the chimney. If the chimney falls through an angle θ relative to the vertical, its center of mass will have gone from a height $y = l/2$ to $y = (l/2)\cos\theta$. The change in the center of mass height Δy_{com} gives a change in gravitational potential energy, which must be equal to the gain in rotational kinetic energy. With I as the moment of inertia of the chimney,

$$mg\Delta y_{com} = \frac{1}{2} mgl\left(1 - \cos\theta\right) = \frac{1}{2} I\omega^2$$

The moment of inertia of the chimney is that of a thin rod of length l and mass m rotating a distance $l/2$ from its center of mass:

$$I = I_{com} + m\left(\frac{l}{2}\right)^2 = \frac{1}{12} ml^2 + \frac{1}{4} ml^2 = \frac{1}{3} ml^2$$

Putting this all together, we can find ω^2, which will give us a_r.

$$\omega^2 = \frac{mgl}{I}\left(1 - \cos\theta\right) = \frac{3g}{l}\left(1 - \cos\theta\right)$$

We are interested in the accelerations at the very end of the chimney, which is thus rotating at a distance l from the base of the rod. The radial acceleration is then

$$a_r = r\omega^2 = l\omega^2 = 3g\left(1 - \cos\theta\right) \approx 5.32\,\mathrm{m/s^2}$$

Since we know ω, we can straightforwardly find α and therefore a_t. It is somewhat easier to implicitly differentiate ω^2 as given above and apply the chain rule:

$$\frac{d\left(\omega^2\right)}{dt} = 2\omega\frac{d\omega}{dt} = 2\omega\alpha = \frac{3g}{l}\sin\theta\frac{d\theta}{dt} = \frac{3g\omega}{l}\sin\theta$$
$$2\omega\alpha = \frac{3g\omega}{l}\sin\theta$$
$$\alpha = \frac{3g}{2l}\sin\theta$$

Given α, we can find a_t:

$$a_t = l\alpha = \frac{3g}{2}\sin\theta \approx 8.44\,\mathrm{m/s^2}$$

Finally, we want to know the angle at which the tangential acceleration equals the gravitational acceleration at the end of the chimney, $a_t = g$. No problem:

$$a_t = g = \frac{3g}{2}\sin\theta \qquad \theta = \sin^{-1}\frac{2}{3} \approx 42°$$

At this point, the end of the chimney is "falling faster than free-fall." At this point, it would be safer to jump off of the chimney ...

The text tells you not to use torques, so obviously we need to show you how to do the problem that way too. It is really not that much harder in the end, and arguably just as straightforward. First: the only torque is due to the weight of the chimney acting at a distance $l/2$ from the center of mass. The force is $\vec{\mathbf{F}} = -mg\,\hat{\mathbf{y}}$, and a vector pointing from the center of rotation to the point of its application is $\vec{\mathbf{r}} = (l/2)\sin\theta\,\hat{\boldsymbol{\imath}} + (l/2)\cos\theta\,\hat{\boldsymbol{\jmath}}$. Or, if you like, the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ is θ, $|\vec{\mathbf{r}}| = l/2$, and $|\vec{\mathbf{F}}| = mg$ since we only need magnitudes. In any case: this torque must give $I\alpha$:

$$I\alpha = \tau_{\text{net}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = |\vec{\mathbf{r}}||\vec{\mathbf{F}}|\sin\theta = -\frac{l}{2}mg\sin\theta$$
$$\tau_{\text{net}} = I\alpha$$

The torque is negative, consistent with its tending to cause a clockwise rotation of the chimney. Using the moment of inertia derived above, you arrive at an expression for α identical to the one above. At this point, you still need ω. You can find this by equating the work done by the torque τ acting through an angle θ to the increase in kinetic energy:

$$W = \int_{\theta_i}^{\theta_f} \tau\, d\theta = \frac{1}{2}I\omega^2$$

The integral is simple, and its limits are from the initial vertical configuration (0) to the final angle of interest (35°). You can verify that this yields the same expression for ω^2 as above.

Two different approaches, same answer, as it has to be. It is sometimes just a matter of taste, memory, and minimizing pain that determines which method to use. I used the torque-work method initially myself, and was a bit confused as to why the book hinted against it.

3. Halliday, Resnick & Walker Problem 11.13

(a) Initially, the bowling ball is purely sliding, and as friction takes hold, the ball begins to roll. During the pure sliding phase, the ball rotates about its center of mass, independent of the overall center of mass motion.
After sufficient time, the rolling motion "catches up" with the sliding motion, and the ball begins to roll - it is no longer spinning about its center of mass, rolling smoothly. This smooth rolling is equivalent to a rotation about a point on the *surface* of the ball (not the center of mass), and as we derived earlier, this means that at the point we have smooth rolling motion, center of mass velocity and angular velocity are simply related:

$$v_{com} = r\omega = 0.11\omega$$

Here $r = 0.11$ m is the given radius of the ball. During the sliding phase, we should write $v_{com} > r\omega$. The angular velocity is not high enough for the ball to "catch" on the lane. [i]

[i] My parents used to own a bowling alley. I can go into much more detail on this problem for the curious.

(b) During the sliding phase, rotation is irrelevant to the dynamics - it is just like any other sliding object we have analyzed. A force of kinetic friction acts at the interface between the ball and the lane, which is equal in magnitude to $f_k = \mu_k F_N$, where μ_k is the coefficient of kinetic friction and $F_N = mg$ the normal force. Since this is the only force acting, we can easily apply Newton's law:

$$\begin{aligned}\sum F &= ma = -f_k \\ a &= -f_k/m = -\mu_k g \approx -2.1\,\mathrm{m/s^2}\end{aligned}$$

(c) The angular acceleration α during the sliding phase is also provided by the friction force f_k. The friction force acts at a distance r from the center of mass, and at a right angle to a radius drawn from the center of mass to the intersection between the ball and lane. Thus, f_k also provides a torque τ, and as the only torque present, it must equal the moment of inertia of the ball times the angular acceleration. Noting $I = \frac{2}{5}mr^2$ for a solid sphere,

$$\begin{aligned}\tau_{\text{net}} &= rf_k = I\alpha = \frac{2}{5}mr^2\alpha \\ \alpha &= \frac{rf_k}{\frac{2}{5}mr^2} = \frac{5\mu_k mg}{2mr} = \frac{5\mu_k g}{2r} \approx 47\,\mathrm{rad/s^2}\end{aligned}$$

(d) During the sliding phase, the rotational and translational motion are essentially decoupled, and we can consider the center of mass motion from the point of view of standard kinematics. That is,

$$v_{com}(t) = v_{com}(0) + at = v_{com}(0) - \mu_k g t$$

Here $v_{com}(0)$ is the initial center of mass velocity, and we imply $t = 0$ at the moment the ball hits the lane. The same is true for the rotational motion, with the added simplification that the initial angular velocity is zero:

$$\omega(t) = \omega(0) + \alpha t = \left(\frac{5\mu_k g}{2r}\right) t$$

Say that the sliding stops at a time t_o. At the moment that sliding stops, we know that $v_{com}(t_o) = r\omega(t_o)$. This yields t_o, the time it takes to stop sliding, in terms of known quantities:

$$\begin{aligned}v_{com}(t_o) &= r\omega(t_o) \\ v_{com}(0) - \mu_k g t_o &= \frac{5}{2}\mu_k g t_o \\ t_o\left(\frac{5}{2}\mu_k g + \mu_k g\right) &= v_{com}(0) \\ t_o &= \frac{v_{com}(0)}{\frac{7}{2}\mu_k g} = \frac{2v_{com}(0)}{7\mu_k g} \approx 1.18\,\mathrm{s}\end{aligned}$$

(e) Given the time to stop sliding, we can also find the distance covered d by standard kinematics:

$$
\begin{aligned}
d &= v_{com}(0)t_o + \frac{1}{2}at_o^2 \\
&= v_{com}(0)\left(\frac{2v_{com}(0)}{7\mu_k g}\right) - \frac{1}{2}\mu_k g\left(\frac{2v_{com}(0)}{7\mu_k g}\right)^2 \\
&= \frac{2\left[v_{com}(0)\right]^2}{7\mu_k g} - \frac{2\left[v_{com}(0)\right]^2}{49\mu_k g} \\
&= \frac{12\left[v_{com}(0)\right]^2}{49\mu_k g} \approx 8.6\,\mathrm{m}
\end{aligned}
$$

(f) The linear (i.e., center of mass) speed at the moment sliding stops is also just kinematics:

$$v_{com}(t_o) = v_{com}(0) + at = v_{com}(0) - \mu_k g t \approx 6.0\,\mathrm{m/s}$$

4. Halliday, Resnick & Walker Problem 13.13

This problem involves a stupid trick, but a very important one that illustrates nicely the power of superposition. The basic idea is this: a solid sphere could be considered as the sum of one with a hole cut out of it, and another piece of the same material the same size as the hole:

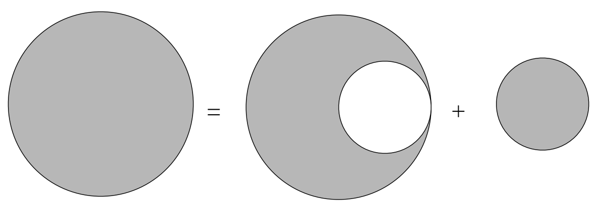

Figure 1: *Stupid tricks with superposition*

The gravitational force of a solid sphere would be the same as that of our object plus a smaller one of the same material (i.e., same density) of half the radius. That means our object is the same as the force of the full sphere *minus* the force of the smaller sphere.

If the whole sphere of radius R has mass M, you can pretty easily show that the sphere of half the radius has mass $M/8$. We can thus consider the force due to the object at hand on a mass m to be that of a whole sphere of mass M a distance d away minus the force of a smaller sphere of mass $M/8$ a distance $d - R/2$ away. Thus,

$$F_{net} = \frac{GMm}{d^2} - \frac{GMm}{8\left(d - R/2\right)^2} = GMm\left(\frac{1}{d^2} - \frac{1}{8\left(d - R/2\right)^2}\right) \approx 8.31 \times 10^{-9}\,\mathrm{N}$$

A very similar electric force problem is here:

`http://faculty.mint.ua.edu/~pleclair/ph106/Homework/HW2_SOLN.pdf`

5. Halliday, Resnick & Walker Problem 13.16

We can consider the rod as a superposition of many tiny point masses of length dx:

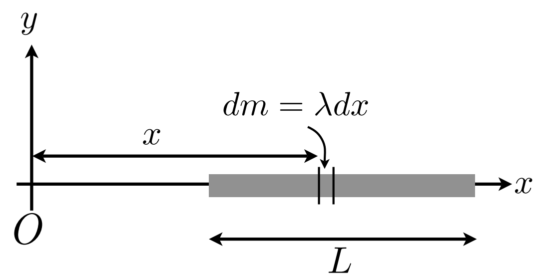

Figure 2: *Stupid tricks with superposition, part II*

If the rod is uniform, of length L and mass M, it has a linear density $\lambda = M/L$, and each bit dx has mass $dm = \lambda\, dx$. The point mass m we which to find the force on is at a distance $x = d$ from the left end of the rod in this scenario. The force from one little bit dm at a distance x from our point mass m is then

$$dF = \frac{Gm\,dm}{x^2} = \frac{Gm\lambda\,dx}{x^2}$$

The total force is found by adding the contributions from all possible dm, which means integrating dx from $x = d$ to $x = d + L$ (i.e., over the length of the rod):

$$F = \int\limits_d^{d+L} \frac{Gm\lambda\,dx}{x^2} = \left.\frac{-Gm\lambda}{x}\right|_L^{d+L} = \frac{-GmM}{L}\left[\frac{1}{d+L} - \frac{1}{d}\right] = \frac{GMm}{d\,(d+L)} \approx 3.0 \times 10^{-10}\,\mathrm{N}$$

Here we reused the definition $\lambda = M/L$ in the third step. Here's a similar electric force problem to find the force between 2 rods:

`http://faculty.mint.ua.edu/~pleclair/ph106/Homework/HW1_SOLN.pdf`

6. Halliday, Resnick & Walker Problem 13.23

We wish for a mass m on the surface to have its gravitational equal its mass times the centripetal acceleration required for circular motion,

$$\sum F = 0 = F_g - ma_c$$

This means that the gravitational force pulling the mass to the surface is exactly providing the centripetal acceleration for circular motion, an the mass feels no net force - the condition for weightlessness on the surface. The gravitational force is readily calculated. With $a_c = v^2/R = mR\omega^2$, where R is the radius of the planet of mass M, and ω is angular velocity. The angular velocity ω is 1 rev/s, or 2π rad/s, thus

$$F_g = \frac{GMm}{R^2} = ma_c = mR\omega^2$$
$$M = \frac{R^3\omega^2}{G} \approx 4.7 \times 10^{24}\,\mathrm{kg}$$

This is approximately 0.8 solar masses, though the rotational speed is outrageously large for a normal star.

7. Halliday, Resnick & Walker Problem 13.68

Let the height of the orbit be h, the mass of each craft m, with the earth's mass and radius M_e and R_e.

(a) The period of the initial orbit T_o is readily found from Kepler's law,

$$T_o = \sqrt{\frac{4\pi^2}{GM} r^3} \approx 5540\,\mathrm{s} \approx 92\,\mathrm{min}$$

(b) The orbital velocity is readily found from conservation of energy, and we have derived it already:

$$v_o = \sqrt{\frac{GM}{R_e + h}} \approx 7680\,\mathrm{m/s}$$

(c) If the new orbital speed is 1% less than the old …

$$v = 0.99 v_o \approx 7600\,\mathrm{m/s}$$

The new kinetic energy is thus

$$K = \frac{1}{2} m v^2 = (0.99)^2 \left(\frac{1}{2} m v_o^2\right) \approx 5.77 \times 10^{10}\,\mathrm{J}$$

(d) Immediately after the 'burn,' the orbital distance has not changed, and therefore neither has the potential energy:

$$U = \frac{-GM_e m}{R_e + h} \approx -1.178 \times 10^{10}\,\mathrm{J}$$

(e) The total energy is just $K + U$:

$$E_{\text{tot}} = K + U \approx -6.0 \times 10^{10}\,\mathrm{J}$$

Note that the total energy is negative, meaning the orbit is bound.

(f) Given the total energy, and knowing the orbit is bound, the most general case is an elliptical orbit. We can relate the total energy to the semi-major axis a of the ellipse, and solve for a

$$E = \frac{-GM_e m}{2a}$$
$$a = \frac{-Gmm}{2E} \approx 6.65 \times 10^6\,\mathrm{m}$$

The height above the earth's surface is of course $a - R_e$.

(g) Originally, Picard was behind by $90\,\mathrm{s}$. His new orbit will be shorter, with period T compared to the original period T_o. That means he will gain $T_o - T$ on the other ship toward making up the $90\,\mathrm{s}$ he was originally behind. Thus, he'll arrive at the same point after

$$t = T_o - T - 90\,\mathrm{s}$$

The new period is readily found from Kepler's law, as in part **(a)**:

$$T = \sqrt{\frac{4\pi^2}{GM}a^3} \approx 5390\,\mathrm{s}$$

Picard gains about 150 s with the new orbit, so if he was originally behind by 90 s, he is now ahead by $t \approx 60\,\mathrm{s}$.

8. Halliday, Resnick & Walker Problem 13.99

Lets do this one in a sneaky way, just for fun. We can start out with force, and it is not that hard, but we have to deal with vectors. Instead, let's find the potential energy first and get the force from that. Let the ring have mass M and radius R, and our point mass m will be a distance x from the center of the ring.

Every bit of mass on the ring is a distance $\sqrt{R^2+x^2}$ from our point mass. Since potential energy is a scalar, each bit of the ring contributes the same potential energy, meaning the ring must be equivalent a single mass M a distance $\sqrt{R^2+x^2}$ away so far as potential energy is concerned. This is *not* true of the force, since force is a vector: each bit of the ring contributes the same *magnitude* of force, but in a different direction. Anyway: the total potential energy is the same as two masses M and m separated by d:

$$U(x) = \frac{-GMm}{\sqrt{R^2+x^2}}$$

When the mass is at the center of the ring, we have simply $U(0) = -GMm/R$. If the mass is released from rest at a distance x from the center of the ring, conservation of energy yields the velocity:

$$\frac{-GMm}{R} + \frac{1}{2}mv^2 = \frac{-GMm}{\sqrt{R^2+x^2}}$$

$$v = \sqrt{2GM\left(\frac{1}{R} - \frac{1}{\sqrt{R^2+x^2}}\right)}$$

What about the force? In general, $\vec{\mathbf{F}} = -\nabla U$ for a conservative force. In this case, the net force is only in the horizontal direction, so way may simply write $F = -dU/dx$. Thus,

$$F = -\frac{dU}{dx} = -\frac{d}{dx}\left(\frac{-GMm}{\sqrt{R^2+x^2}}\right) = GMm\left(\frac{(2x)\left(\frac{-1}{2}\right)}{\left(R^2+x^2\right)^{3/2}}\right) = \frac{-GMmx}{\left(R^2+x^2\right)^{3/2}}$$

It is easy enough to do the problem in the usual way - break the ring up into bits dm and integrate, and in fact the integral is trivial in this case. In general, however, finding the potential first and then getting the force is a nice trick for solving problems. Potential is a scalar, which makes it much easier than dealing with vector forces, and you only have to differentiate to boot. This trick will show up a lot in PH126, where the problems can be otherwise nearly intractable.

9. An object of mass m is dropped from a height h above the surface of a planet of mass M and radius R. Assume the planet has no atmosphere so that friction can be ignored. Further assume the planet has no life that may be harmed by subsequent portions of this problem.

(a) What is the speed of the mass just before it strikes the surface of the planet? Do *not* assume that h is small compared with R.

(b) Show that the expression from **(a)** reduces to $v = \sqrt{2gh}$ for $h \ll R$.

(c) How long does it take for the object to fall to the surface for an arbitrary value of h? Use any means necessary to evaluate the integral required. Bonus points (10%) for code submissions.