University of Alabama
Department of Physics and Astronomy

PH 125 / LeClair Spring 2009

Problem Set 11: Solutions

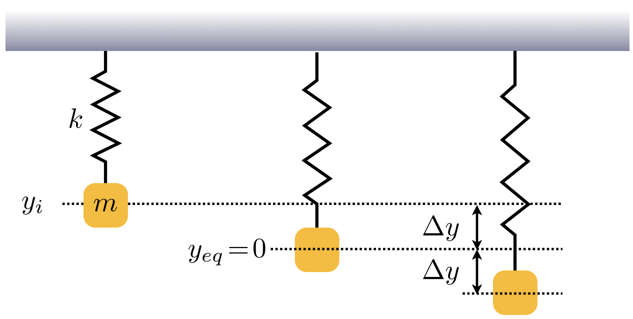
1. Halliday, Resnick & Walker Problem 15.26, two springs in series.

See `http://scienceworld.wolfram.com/physics/SpringsTwoSpringsinSeries.html` for a good solution to this problem. With the numbers given, you should find

$$f = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{(k_1+k_2)\,m}} \approx 18\,\text{Hz}$$

2. Halliday, Resnick & Walker Problem 15.37: mass hanging from a spring.

Our mass starts out at position y_i, corresponding to the un-stretched length of the spring. When released, its lowest position is $2\Delta y = 10\,\text{cm}$ below y_i during the subsequent oscillations. This means that the amplitude of the simple harmonic motion is Δy, symmetric about an equilibrium position y_{eq} – both y_i and the lowest point in the motion are Δy from y_{eq}. For convenience, let the equilibrium position be our origin, $y_{eq} = 0$, with the $\hat{\jmath}$ direction being upward. With this choice, $y_i = \Delta y$ is the amplitude of harmonic motion. Make use of the figure below.



(a) We can find the frequency of oscillation by considering the forces acting on the mass, which are only gravity and the spring restoring force. If the mass moves a distance y from equilibrium,

$$ma = mg - ky$$

At the equilibrium position, the string is stretched by an amount Δy compared to its natural length, and $a = 0$:

$$mg = k\Delta y \quad\Longrightarrow\quad \frac{k}{m} = \frac{g}{\Delta y}$$

In principle, we can now must use $f=(2\pi)^{-1}\sqrt{k/m}$ to find the frequency of oscillation. However, should we be concerned whether our solution to simple harmonic motion is valid in the presence of an additional constant force (i.e., gravity)? Our force balance equation, suitably rearranged, reads

$$\frac{d^2y}{dt^2}+\frac{k}{m}y-g=0$$

Without the additional constant gravitational acceleration, we would have our equation for simple harmonic motion. A simple substitution will recover the usual equation for simple harmonic motion, for which we know the solution. Let $y'=y-mg/k$, which gives $d^2y'/dt^2=d^2y/dt^2$. Making the substitution in our equation above,

$$\frac{d^2y'}{dt^2}+\frac{k}{m}y'+g-g=\frac{d^2y'}{dt^2}+\frac{k}{m}y'=0$$

We have recovered the standard equation of motion for a simple harmonic oscillator, and thus the presence of an additional constant force serves only to shift the origin by an amount mg/k. This shift leaves the frequency of oscillation unchanged at $f=(2\pi)^{-1}\sqrt{k/m}$. The substitution we made physically corresponds to shifting the equilibrium position downward by an amount mg/k – exactly how far the mass pulls the spring down once it is attached. This shift is just a choice of origin so far as the equations are concerned, the physics is unchanged. In the end, we are justified in using our beloved equations of simple harmonic motion, so long as we choose our origin at the new equilibrium position y_i-mg/k, which we have already done!

With our now-justified solution the numbers given,

$$f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}=\frac{1}{2\pi}\sqrt{\frac{g}{\Delta y}}\approx 2.2\,\mathrm{Hz}$$

Recall that units of s^{-1} are commonly called *Hertz*, abbreviated Hz.

(b) When the mass is 8 cm below its initial position, what is its speed? There are several ways to go about this.

Conservation of Energy: First, and perhaps most straightforwardly, we can use conservation of mechanical energy. Let the position of interest at 8 cm be y_f. At the starting position of the mass, y_i, we have only the gravitational potential energy of the mass, since the mass is at rest and the spring is un-stretched. At position y_f, the mechanical energy consists of three parts: the new gravitational potential energy, the kinetic energy of the mass, and the potential energy of the now stretched spring. For the latter term, it is key to remember that the spring has been stretched by an amount y_i-y_f, since it started at its un-stretched length at y_i.[i] Writing down all the requisite energy terms, it is no big trick to solve for v

$$\begin{aligned}
mgy_I &= mgy_f+\frac{1}{2}mv^2+\frac{1}{2}k\left(y_i-y_f\right)^2\\
\frac{1}{2}mv^2 &= mg\left(y_i-y_f\right)-\frac{1}{2}k\left(y_i-y_f\right)^2\\
v^2 &= 2g\left(y_i-y_f\right)-\frac{k}{m}\left(y_i-y_f\right)^2 \qquad \left(\text{note}\quad \frac{k}{m}=\frac{g}{\Delta y}\right)\\
v &= \sqrt{2g\left(y_i-y_f\right)+\frac{g}{\Delta y}\left(y_i-y_f\right)^2}
\end{aligned}$$

[i] Be careful that in the present case the equilibrium position is *not* the un-stretched position, and therefore not the position of zero spring potential energy.

Noting that we are told $y_i - y_f = 8$ cm and $\Delta y = 5$ cm (and converting everything to meters),

$$v = \pm 0.56\,\text{m/s}$$

The $\pm$ in this case is physically meaningful – at 8 cm below the starting position, the mass can be going either upward or downward with the same speed.

Equation of Motion: Since we have established that our hanging mass follows simple harmonic motion, we know the general solution for $y(t)$:

$$y(t) = A\cos\omega t + B\sin\omega t$$

From $y(t)$, we can readily find $v = dy/dt$, we need only find the time at which $y(t)$ corresponds to the given position. If the mass starts at rest at y_i, our boundary conditions are $y(0) = 0$ and $v(0) = 0$. Our general solution then becomes

$$y(t) = y_i \cos\omega t$$

in order to be consistent with our boundary conditions. At what time does $y(t)$ correspond to the point of interest? The mass starts out at $y_i = 5$ cm above equilibrium. That means that the position of interest, 8 cm below y_i, is then 3 cm *below* equilibrium. Thus, we are interested in the time t_o such that $y(t_o) = -3 \equiv y_f$ (since $\hat{\jmath}$ is upward).

$$\begin{aligned} y_f &= y_i \cos\omega t_o \\ t_o &= \frac{1}{\omega}\cos^{-1}\left[\frac{y_f}{y_i}\right] \end{aligned}$$

The velocity is now easily found:

$$\begin{aligned} v(t) &= \frac{dy}{dt} = -\omega y_i \sin\omega t \\ v(t_o) &= -\omega y_i \sin\left[\cos^{-1}\left(\frac{y_f}{y_i}\right)\right] = -\omega y_i\sqrt{1-\left(\frac{y_f}{y_i}\right)^2} = -\omega\sqrt{y_i^2 - y_f^2} \\ &= -2\pi f\sqrt{y_i^2 - y_f^2} \approx 0.56\,\text{m/s} \end{aligned}$$

Note that we used the identity $\sin\left[\cos^{-1} x\right] = \sqrt{1-x^2}$ here. Also note that this is simply the equation of an ellipse, which leads us to our next method …

Phase space relationships: As we discussed in class, for the general simple harmonic motion solution

$$y(t) = C\cos(\omega t + \delta)$$

The allowed values of position y and momentum p for our oscillator satisfy the equation of an ellipse:

$$\frac{y^2}{C^2} + \frac{p^2}{m^2\omega^2 C^2} = 1$$

That is, position and momentum are conjugate variables, and their values are linked. Since we know the position of interest y, there are at most two possible momenta, which will differ only by a sign. Noting that in the present case our boundary conditions give $C = y_i$, and using $p = mv$

$$\begin{aligned}
1 - \frac{y^2}{y_i^2} &= \frac{m^2 v^2}{m^2 \omega^2 y_i^2} = \frac{v^2}{\omega^2 y_i^2} \\
v^2 &= \omega^2 y_i^2 \left(1 - \frac{y^2}{y_i^2}\right) = \omega^2 \left(y_i^2 - y_f^2\right) \\
v &= \pm\omega\sqrt{y_i^2 - y_f^2} = \pm 2\pi f \sqrt{y_i^2 - y_f^2}
\end{aligned}$$

Precisely the same solution, quite a bit faster.

(c) We are told that the addition of a $0.3\,\text{kg}$ mass halves the frequency of oscillation. If the original mass is m_1, and the new mass is $m_2 = 0.3\,\text{kg}$, the original frequency is

$$f_o = \sqrt{\frac{k}{m_1}}$$

The new frequency is determined by the total mass, now $m_1 + m_2$:

$$f = \frac{1}{2} f_o = \sqrt{\frac{k}{m_1 + m_2}}$$

Combining, and solving for m_1,

$$\begin{aligned}
\frac{1}{2} f_o &= \frac{1}{2}\sqrt{\frac{k}{m_1}} = \sqrt{\frac{k}{m_1 + m_2}} \\
\frac{k}{4m_1} &= \frac{k}{m_1 + m_2} \\
k\left(m_1 + m_2\right) &= 4km_1 \\
3m_1 &= m_2 \quad \Longrightarrow \quad m_1 = 0.1\,\text{kg}
\end{aligned}$$

(d) The new equilibrium position is found just like the original equilibrium position: the total weight balances the spring's restoring force. Let the new equilibrium position be a distance y'_{eq} below the original equilibrium:

$$\begin{aligned}
ky'_{eq} &= \left(m_1 + m_2\right) g \\
y'_{eq} &= \frac{g}{k}\left(m_1 + m_2\right) = \frac{g\Delta y}{m_1 g}\left(m_1 + m_2\right) \qquad \left(\text{note} \quad k = \frac{m_1 g}{\Delta y}\right) \\
&= \left(\frac{m_1 + m_2}{m_1}\right)\Delta y = 4\Delta y \qquad \left(\text{note} \quad 3m_1 = m_2\right) \\
&\approx 0.2\,\text{m}
\end{aligned}$$

3. Halliday, Resnick & Walker Problem 15.55

In the end, we only have a physical pendulum, and the period is given by

$$T = 2\pi\sqrt{\frac{I}{mgh}}$$

where I is the moment of inertia of the rod (of mass m) about the pivot point, and h is the distance between the rod's center of mass and the pivot point. Let the pivot be a distance x from the end of the rod, making it a distance $l/2-x$ from the center of mass. The moment of inertia is then

$$I = I_{com} + m\left(\frac{l}{2} - x\right)^2 = \frac{1}{12}ml^2 + m\left(\frac{l}{2} - x\right)^2$$

The distance between the center of mass and the pivot is $h=l/2-x$, so

$$I = \frac{1}{12}ml^2 + mh^2$$

The period is thus

$$T = 2\pi\sqrt{\frac{\frac{1}{12}l^2 + h^2}{gh}} = 2\pi\sqrt{\frac{l^2}{12gh} + \frac{h}{g}}$$

We wish to find x such that T is a maximum, which means $dT/dx=0$. Noting that $dT/dx=-dT/dh$,

$$\begin{aligned}
\frac{dT}{dx} &= -\frac{dT}{dh} = 0 \\
\frac{d}{dh}\left[2\pi\sqrt{\frac{\frac{1}{12}l^2 + h^2}{gh}}\right] &= 0 \\
2\pi\left(\frac{1}{2}\right)\left(\frac{-l^2}{12gh^2} + \frac{1}{g}\right)\left(\frac{\frac{1}{12}l^2 + h^2}{gh}\right)^{-1/2} &= 0 \\
\implies \quad \frac{-l^2}{12gh^2} + \frac{1}{g} &= 0 \\
12h^2 &= l^2 \\
h &= \frac{l}{2\sqrt{3}} \approx 0.29l
\end{aligned}$$

A quick second derivative test or a plot of dT/dh verifies that this is indeed a minimum, not a maximum. The minimum period is therefore

$$T_{\text{min}} = T\Big|_{h=\frac{l}{2\sqrt{3}}} = 2\pi\sqrt{\frac{\frac{1}{12}l^2 + \frac{1}{12}l^2}{g\frac{l}{2\sqrt{3}}}} = 2\pi\sqrt{\frac{l}{\sqrt{3}g}} \approx 2.26\,\text{s}$$

(b) Given $T \propto \sqrt{l}$, the period increases as l increases.

(c) The period is independent of m, and remains unchanged as m increases.

4. Halliday, Resnick & Walker Problem 15.63

Resonance - maximum amplitude of oscillation in this case - will occur when the frequency at which the car hits successive bumps in the washboard. After hitting a bump, the car will bounce up, come back down, and reach a minimum position. If the next bump comes at exactly the moment at which the car is at its minimum vertical position, the 'push' from the next bump will be maximally efficient, leading to the largest amplitude of vibration.

If the distance between bumps is d, the time between bumps traveling at constant speed v_o is

$$T_o = \frac{d}{v_o}$$

For resonance to occur, this time interval must be the same as the period of simple harmonic motion. If the total mass of the car and passengers is m_{tot} and the shocks have an effective spring constant k,

$$T_o = 2\pi\sqrt{\frac{m_{\text{tot}}}{k}} = \frac{d}{v_o}$$
$$k = \frac{4\pi^2 m_{\text{tot}} v_o^2}{d^2}$$

The total mass of the car plus four passengers is $m_{\text{tot}} = m_{\text{car}} + 4m_{\text{p}}$. If the four passengers get out, the difference in the distance the spring is compressed Δx must be related to the weight of the four passengers:

$$k\Delta x = 4m_p g$$

That is, if the change in weight is $4m_p g$, there must be an equivalent change in the spring's restoring force to reach a new equilibrium. Solving for Δx and substituting our expression for k,

$$\Delta x = \frac{4m_p g}{k} = \frac{4m_p g d^2}{4\pi^2 m_{\text{tot}} v_o^2} = \frac{m_p g d^2}{\pi^2 \left(m_c + 4m_p\right) v_o^2} \approx 5.0\,\text{cm}$$

5. Halliday, Resnick & Walker Problem 15.106

The mechanical energy in this case consists of rotational kinetic energy, translational kinetic energy, and potential energy stored in the spring. Let $x = 0$ correspond to the un-stretched length of the spring, which is also the equilibrium position of this system. The total mechanical energy is

$$E_{tot} = K + U = K_t + K_r + U_s = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2$$

Before we proceed, one aside: if a circular object is rolling smoothly, the frictional force plays no roll - essentially, at each instant in time it is a different bit of the circular surface contacting the ground. Friction only imparts a retarding force and dissipates energy from the system when there is sliding involved. See `http://webphysics.davidson.edu/faculty/dmb/py430/friction/rolling.html` for a good explanation. Basically, pure rolling involves no work done by friction, so we are justified in writing the total energy as we have above.

In the case of pure rolling, we can relate the linear velocity of the center of mass v and the angular velocity ω through the radius of the cylinder, $v = r\omega$. Substituting for ω above, and noting $I = kmr^2$ in general (with $k = 1/2$ in the present case)

$$
\begin{aligned}
E_{tot} &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 \\
&= \frac{1}{2}mv^2 + \frac{1}{2}\left(kmr^2\right)\left(\frac{v}{r}\right)^2 + \frac{1}{2}kx^2 \\
&= \left(\frac{k+1}{2}\right)mv^2 + \frac{1}{2}kx^2 \\
&= (k+1)\,K_t + \frac{1}{2}kx^2
\end{aligned}
$$

We are told the maximum displacement is $x_{\text{max}} = \frac{1}{4}\,\text{m}$. At maximum displacement, both kinetic terms are zero, and the energy is purely potential:

$$
\frac{1}{2}kx_{\text{max}}^2 = E_{\text{tot}} = \frac{3}{32}\,\text{J}
$$

On the other hand, at the equilibrium position, the energy is entirely kinetic. Since the only relevant forces are conservative (having established friction plays no role), mechanical energy is conserved, at equilibrium

$$
\begin{aligned}
K_r + K_t &= (k+1)\,K_t = E_{\text{tot}} \qquad \left(k = \frac{1}{2}\right) \\
\Longrightarrow K_t &= \frac{E_{\text{tot}}}{k+1} = \frac{1}{16}\,\text{J} \\
K_r &= E_{\text{tot}} - K_t = \frac{1}{32}\,\text{J}
\end{aligned}
$$

In order to find the period of motion, we would like to find $a = d^2x/dt^2$ and show that it is proportional to position, $a = -\omega^2 x$. We could write down a force and torque balance and arrive at the solution without an inordinate amount of work. However, there is an easier way.

We can also find the period by noting that $dE/dT = 0$, since mechanical energy is conserved. Taking the time derivative of the total energy will give us factors of acceleration and position; if we are lucky, that is all.

$$
\begin{aligned}
\frac{dE_{tot}}{dt} &= \frac{d}{dt}\left[\left(\frac{k+1}{2}\right)mv^2 + \frac{1}{2}kx^2\right] = 0 \\
0 &= \left(\frac{k+1}{2}\right)m\,(2v)\left(\frac{dv}{dt}\right) + \frac{1}{2}k\,(2x)\left(\frac{dx}{dt}\right) \\
0 &= (k+1)\,mva + kxv \\
0 &= (k+1)\,ma + kx \qquad (v \neq 0) \\
a &= -\frac{k}{m\,(k+1)} \equiv -\omega^2 x
\end{aligned}
$$

This is just the usual equation for simple harmonic motion, for which we know the solution

$$
\begin{aligned}
\omega &= \sqrt{\frac{k}{m\,(k+1)}} \\
T &= \frac{2\pi}{\omega}
\end{aligned}
$$

Using $k = 1/2$,

$$\omega = \sqrt{\frac{2k}{3m}}$$
$$T = 2\pi\sqrt{\frac{3m}{2k}}$$

The division by v above means that this solution is not valid at the turning points, where $v = 0$, which is not really a restriction at all.