UNIVERSITY OF ALABAMA
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Problem Set 12 Solutions

1. Halliday, Resnick & Walker Problem 15.24. Two blocks ($m=1.8\,\text{kg}$ and $M=10\,\text{kg}$) and a spring ($k=200\,\text{N/m}$) are arranged on a horizontal, frictionless surface. The coefficient of static friction between the two blocks is 0.40. What amplitude of simple harmonic motion of the the spring-blocks system puts the smaller block on the verge of slipping over the larger block?

If the upper block m is on the verge of slipping, it means that the force exerted on it by the larger block equals the maximum force of static friction $f_{s,\text{max}}=\mu_s mg$. The force exerted on the smaller block is due to the acceleration of the larger block, which we know to be $a=\omega^2 x_m$ during simple harmonic motion. If the acceleration a exceeds $f_{s,\text{max}}/m$, the smaller block will fall off. The angular frequency of simple harmonic motion ω is readily found by noting that the spring k is connected to a total mass $M+m$

$$\omega=\sqrt{\frac{k}{M+m}}$$

The amplitude of simple harmonic motion gives us a maximal acceleration, which we can compare with $f_{s,\text{max}}/m$ - the latter must be larger to avoid the smaller block falling off.

$$\begin{aligned}
\frac{f_{s,\text{max}}}{m} &> \omega^2 x_m \\
\mu_s g &> \frac{k x_m}{M+m} \\
x_m &< \frac{\mu_s g\,(M+m)}{k} \approx 0.23\,\text{m}
\end{aligned}$$

2. Halliday, Resnick & Walker Problem 15.56. A $2.50\,\text{kg}$ disk of diameter $D=42.0\,\text{cm}$ is supported by a rod of length $L=76.0\,\text{cm}$ and negligible mass that is pivoted at its end. **(a)**With the massless torsion spring unconnected, what is the period of oscillation? **(b)** With the torsion spring connected, the rod is vertical at equilibrium. What is the torsion constant of the spring if the period of oscillation has been decreased by $0.500\,\text{s}$?

(a) Without the torsion spring, this is just a physical pendulum. In order to find its period, we need only the moment of inertia of the pendulum bob about the pivot point of the pendulum. The bob is a simple disk, with $I_{com}=\frac{1}{2}mr^2$, and it rotates about the pendulum's pivot point, a distance $r+L$ away (here $r=D/2$ and m is the mass of the disk). The moment of inertia is then found from the parallel axis theorem:

$$I=\frac{1}{2}mr^2+m\,(r+L)^2$$

We have already derived the formula for the period of a physical pendulum (T_o), given the moment of inertia I and the distance from the bob's center of mass to the pivot point h:

$$T_o = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{\frac{1}{2}mr^2 + m\left(r+L\right)^2}{mg\left(r+L\right)}} = 2\pi\sqrt{\frac{\frac{1}{2}r^2 + \left(r+L\right)^2}{g\left(r+L\right)}} \approx 2.00\,\mathrm{s}$$

(b) If the pendulum has a shorter period when the torsion spring is connected, this must mean that the restoring torque due to the spring acts in the same direction as gravity. We can find the period of the new pendulum by considering both torques together, and noting that the sum of all torques must give I times the angular acceleration.

If our pendulum is inclined at an angle θ relative to its vertical equilibrium position, the magnitude of the torque due to a torsion spring is $\kappa\theta$, while the torque on the pendulum due to the weight of the bob is $mgh\sin\theta$. The torque balance then reads (noting that we have "restoring torques" present to get the signs right)

$$\sum \tau = -mgh\sin\theta - \kappa\theta = I\frac{d^2\theta}{dt^2}$$

If we assume small deviations from equilibrium (small θ), then $\sin\theta \approx \theta$, we recover our equation for simple harmonic motion:

$$\begin{aligned}
-mgh\sin\theta - \kappa\theta \approx -\left(mgh+\kappa\right)\theta &= I\frac{d^2\theta}{dt^2} \\
\frac{d^2\theta}{dt^2} &= -\left(\frac{mgh+\kappa}{I}\right)\theta \\
\implies \quad \omega &= \sqrt{\frac{mgh+\kappa}{I}} \\
T &= 2\pi\sqrt{\frac{I}{mgh+\kappa}}
\end{aligned}$$

We know the new period T with the torsion spring is 0.500 sec shorter than T_o, $T-T_o$=-0.500 s, so we know enough to find κ

$$\begin{aligned}
T &= 2\pi\sqrt{\frac{I}{mgh+\kappa}} \\
\left(\frac{2\pi}{T}\right)^2 &= \frac{mgh+\kappa}{I} \\
\kappa &= I\left(\frac{2\pi}{T}\right)^2 - mgh = I\left(\frac{2\pi}{T_o - 0.500}\right)^2 - mgh \approx 18.4\,\mathrm{N\,m/rad}
\end{aligned}$$

3. Halliday, Resnick & Walker Problem 15.111. The center of oscillation ...

(a) The net horizontal force is F, since the batter is assumed to exert no horizontal force on the bat. The horizontal acceleration must then be $a = F/m$, so long as F is acting.

(b) The only torque on the system is due to F acting at P, a distance $\frac{2}{3}L$ from the end of the bat (O). If P is $\frac{2}{3}L$ from O, then it is $\left[\frac{2}{3} - \frac{1}{2}\right]L = \frac{1}{6}L$ from the center of mass. The torque about the center of mass is found from this distance and the magnitude of F, and must equal the bat's moment of inertia about its end point times the angular acceleration:

$$\sum \tau = \left[\frac{2}{3} - \frac{1}{2}\right] LF = \frac{1}{6}LF = I\alpha = \frac{1}{12}mL^2\alpha$$
$$\alpha = \frac{2F}{ml}$$

(c) The angular acceleration about the center of mass results in a tangential linear acceleration of αr, where r is the distance between the center of mass and the end of the bat, $r = L/2$. This acts in the opposite direction of the linear acceleration provided by F, so the net linear acceleration is

$$a = \frac{F}{m} - \frac{1}{2}L\alpha = \frac{F}{m} - \frac{F}{m} = 0$$

The point P is thus the special point where a horizontal impact will result in no net acceleration. From the Wikipedia:

> The center of percussion is the point on an object where a perpendicular impact will produce translational and rotational forces which perfectly cancel each other out at some given pivot point, so that the pivot will not be moving momentarily after the impulse. The same point is called the center of oscillation for the object suspended from the pivot as a pendulum. - Wikipedia, "Center of Oscillation"

(d) Once again we refer you to the Wikipedia,

> The term originally referred to various pieces of sporting equipment, notably cricket and baseball bats and tennis racquets. When hitting the ball, the bat (for instance) will rebound, but there is a location along the bat where this force is completely balanced out by turning force of the bat. If the ball is hit closer to the end of the bat, the grip of the bat will try to rotate forward out of the batter's hands, whereas if the batter hits it closer to the handle, the bat's tip will try to rotate forward. There is a small "sweet spot" where these two tendencies cancel out. The "sweet spot" location on a given baseball bat varies however it is approximately 6-1/2" from the end of the barrel.
>
> Although the sweet spot gives a powerful and clean hit, peak ball speed occurs nearer the tip of the bat where the bat is travelling at greater speed. - Wikipedia, "Sweet spot"

4. Halliday, Resnick & Walker Problem 16.25. A uniform rope of mass m and length L hangs from a ceiling.

(a) The speed of transverse wave as a function of the position y from the bottom of the rope depends only on the tension at that point. You can't push on a rope, so the tension at that point depends only on how much rope is hanging below that point, a length y. If we define a linear density $\mu = m/L$ - which we can do since the rope is uniform - the weight of the rope below point y is $y\mu$. The tension in the rope at that point is then just $y\mu g$, and the wave speed is

$$v(y) = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{y\mu g}{\mu}\mu} = \sqrt{yg}$$

(b) The time a wave takes to travel the distance of the rope is found by noting that $v(y) = dy/dt$ and integrating along the length of the rope:

$$
\begin{aligned}
v(y) &= \frac{dy}{dt} \\
dt &= \frac{dy}{v(y)} \\
\Delta t &= \int_0^L \frac{dy}{v(y)} = \int_0^L \sqrt{\frac{1}{yg}}\,dy = 2\sqrt{\frac{y}{g}}\Bigg|_0^L = 2\sqrt{\frac{L}{g}}
\end{aligned}
$$

5. Halliday, Resnick & Walker Problem 16.34. A sinusoidal wave of angular frequency $\omega = 1200\,\text{rad/s}$ and amplitude $3.00\,\text{mm}$ is sent along a cord with linear density $2.00\,\text{g/m}$ and tension $1200\,\text{N}$.

(a) The average rate of energy transfer can be found from the angular frequency ω, amplitude y_m, linear density μ, and tension T if we note that the wave speed is $v = \sqrt{T/\mu}$:

$$
\mathscr{P}_{\text{avg}} = \frac{1}{2}\mu v \omega^2 y_m^2 = \frac{1}{2}\mu\left(\sqrt{\frac{T}{\mu}}\right)\omega^2 y_m^2 = \frac{1}{2}\omega^2 y_m^2\sqrt{T\mu} \approx 10\,\text{W}
$$

(b) Two strings, twice as much power ...the waves cannot interfere if they travel on separate strings 20 W.

(c) Now the waves are along the same string, and we must consider interference. If the two waves have a phase difference φ, they sum to form a new wave whose *amplitude* is given by $2y_m \cos\frac{\varphi}{2}$. The power transmitted by this resultant wave is found by replacing the amplitude of a single wave y_m in our power equation above with $2y_m \cos\frac{\varphi}{2}$:

$$
\mathscr{P}'_{\text{avg}} = \frac{1}{2}\omega^2 y_m^2\sqrt{T\mu}\left[4\cos^2\frac{\varphi}{2}\right] = \left[4\cos^2\frac{\varphi}{2}\right]\mathscr{P}_{\text{avg}}
$$

In the first case, we have $\varphi = 0$, and since $4\cos^2\frac{\varphi}{2} = 4$ we have four times our previous power, $\mathscr{P}'_{\text{avg}} \approx 40\,\text{W}$.

(d) With $\varphi = 0.4\pi$, $4\cos^2\frac{\varphi}{2} \approx 2.62$, and thus $\mathscr{P}'_{\text{avg}} \approx 26.2\,\text{W}$.

(e) With $\varphi = \pi$, we have perfect destructive interference: $\cos^2\frac{\varphi}{2} = 0$ and thus $\mathscr{P}'_{\text{avg}} = 0$.

6. Halliday, Resnick & Walker Problem 16.35. Two sinusoidal waves of the same frequency travel in the same direction along a string. If $y_{m1} = 3.0\,\text{cm}$, $y_{m2} = 3.0\,\text{cm}$, $\varphi_1 = 0$, and $\varphi_2 = \pi/2\,\text{rad}$, what is the amplitude of the resultant wave?

The overall amplitude is most easily found by writing both waves as complex exponentials, $y = y_m e^{i\theta}$. Since $\varphi_1 = 0$, let $\varphi = \varphi_2 = \pi/2\,\text{rad}$.[i]

$$
\begin{aligned}
y_1(x,t) &= y_{m1} e^{i(kx-\omega t)} \\
y_2(x,t) &= y_{m2} e^{i(kx-\omega t+\varphi)} = y_{m2} e^{i(kx-\omega t)} e^{i\varphi}
\end{aligned}
$$

Written in this way, we can manipulate waves as vectors in polar (r, θ) coordinates - the amplitudes are r, and the

[i]In general, we only need to worry about the phase *difference* between the two waves, since we can always define one wave to have zero phase by appropriately choosing our origin or $t = 0$.

complex exponentials are the “angles." Adding the two waves is the same as adding two vectors of lengths y_{m1} and y_{m2} which make an angle φ with each other. Since in this case $\varphi = \pi/2\,\text{rad}$, we really just have to find the hypotenuse of a right triangle:

$$y_t = \sqrt{y_{m1}^2 + y_{m2}^2} = 5\,\text{cm}$$

7. Halliday, Resnick & Walker Problem 16.60. A string tied to a sinusoidal oscillator ...

The modes of a standing wave on a string of length L are determined only by the mode number n, the length of the string L, and the wave speed v. The wave speed can be determined from the tension in the string T and its linear density μ:

$$f = \frac{nv}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

What we have are two different modes n and m which correspond to different tensions T_1 and T_2. In each case, the tension is provided by the hanging mass, so $T_1 = m_1 g$ and $T_2 = m_2 g$. We do not have enough information to so proceed based solely on the equation above and either single mass.

We are told that masses m_1 and m_2 result in standing wave patterns, but *no masses in between* m_1 *and* m_2. This can only be the case if the two masses result in standing wave patterns one mode apart, which would mean that there are no stable wave patterns for masses in between m_1 and m_2. The mode indices must then be adjacent integers. Let the lower mode be n, and the higher $n+1$. From the equation above, the higher mode $n+1$ must correspond to the smaller mass m_1 if f is to be constant.

The frequency of oscillation f is fixed by the oscillator when either mass is present, and thus

$$\begin{aligned}
f = \frac{n}{2L}\sqrt{\frac{m_2 g}{\mu}} &= \frac{n+1}{2L}\sqrt{\frac{m_1 g}{\mu}} \\
n\sqrt{m_2} &= (n+1)\sqrt{m_1} \\
n\left(\sqrt{m_2} - \sqrt{m_1}\right) &= \sqrt{m_1} \\
n &= \frac{\sqrt{m_1}}{\sqrt{m_2} - \sqrt{m_1}} = 4
\end{aligned}$$

The standing wave modes of the string must be the fourth and fifth modes. Now we may rearrange our first equation to solve for μ. Plugging in $n = 4$ and $m_2 = 0.447\,\text{kg}$,

$$\mu = \frac{n^2 m_2 g}{4 f^2 L^2} \approx 8.46 \times 10^{-4}\,\text{kg/m}$$