University of Alabama
Department of Physics and Astronomy

PH 125 / LeClair Spring 2009

# Problem Set 1: Solutions 1

## Problems due 13 January 2009

**1.** Water is poured into a container that has a leak. The mass $m$ of the water is as a function of time $t$ is

$$m = 5.00t^{0.8} - 3.00t + 20.00$$

with $t \geq 0$, $m$ in grams, and $t$ in seconds. At what time is the water mass greatest?

**Given:** Water mass versus time $m(t)$.

**Find:** The time $t$ at which the water mass $m$ is greatest. This can be accomplished by finding the time derivative of $m(t)$ and setting it equal to zero, followed by checking the second derivative to be sure we have found a maximum.

**Sketch:** It is useful to plot the function $m(t)$ and graphically estimate about where the maximum should be, roughly.[i]

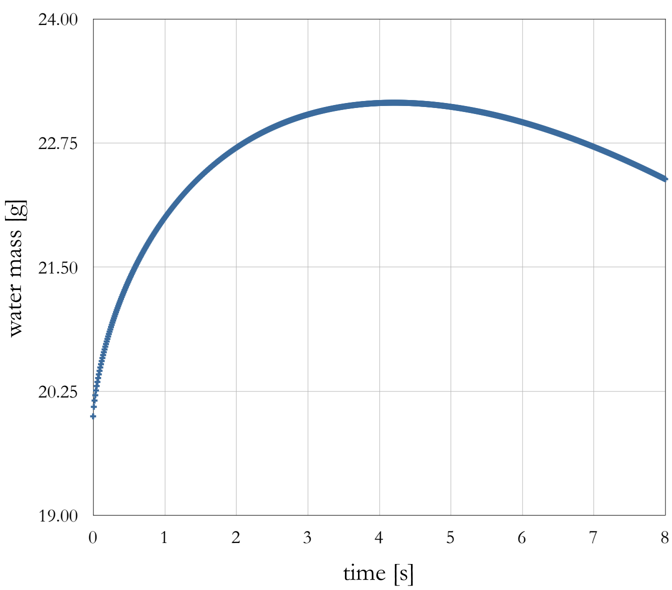


**Figure 1:** *Water mass versus time, problem 1. Note the rather expanded vertical axis, with offset origin.*

It is clear that there is indeed a maximum water mass, and it occurs just after $t = 4$ s.

[i] It is relatively easy to do this on a graphing calculator, which can be found online these days: `http://www.coolmath.com/graphit/`.

**Relevant equations:** We need to find the maximum of $m(t)$. Therefore, we need to set the first derivative equal to zero. We must also check that the second derivative is negative to ensure that we have found a maximum, not a minimum. Therefore, only two equations are needed:

$$\frac{dm}{dt} = \frac{d}{dt}\left[m(t)\right] = 0 \qquad \text{and} \qquad \frac{d^2m}{dt^2} = \frac{d^2}{dt^2}\left[m(t)\right] < 0 \quad \Longrightarrow \quad \text{maximum in } m(t)$$

**Symbolic solution:**

$$\begin{aligned}
\frac{dm}{dt} &= \frac{d}{dt}\left[5t^{0.8} - 3t + 20\right] = 0.8\left(5t^{0.8-1}\right) - 3 = 4t^{-0.2} - 3 = 0 \\
4t^{-0.2} - 3 &= 0 \\
t^{-0.2} &= \frac{3}{4} \\
\Longrightarrow \quad t &= \left(\frac{3}{4}\right)^{-5} = \left(\frac{4}{3}\right)^{5}
\end{aligned}$$

Thus, $m(t)$ takes on an extreme value at $t = (4/3)^5$. We did not prove whether it is a maximum or a minimum however! This is important ... so we should apply the *second derivative* test.

Recall briefly that after finding the extreme point of a function $f(x)$ via $df/dx|_{x=a} = 0$, one should calculate $d^2f/dx^2|_{x=a}$: if $d^2f/dx^2|_{x=a} < 0$, you have a maximum, if $d^2f/dx^2|_{x=a} > 0$ you have a minimum, and if $d^2f/dx^2|_{x=a} = 0$, the test basically wasted your time. Anyway:

$$\begin{aligned}
\frac{d^2m}{dt^2} &= \frac{d}{dt}\left[\frac{dm}{dt}\right] = \frac{d}{dt}\left[4t^{-0.2} - 3\right] = -0.2\left(4t^{-0.2-1}\right) = -0.8t^{-1.2} \\
\frac{d^2m}{dt^2} &< 0 \quad \forall \quad t > 0
\end{aligned}$$

$\frac{d^2m}{dt^2}$ is greater than zero[ii] $t > 0$, since $t^{-1.2}$ is always positive in that regime, which means we have indeed found a maximum.

**Numeric solution:** Evaluating our answer numerically, remembering that $t$ has units of seconds (s):

$$t = \left(\frac{4}{3}\right)^5 \approx 4.21399 \xrightarrow[\text{digits}]{\text{sign.}} 4.21\,\text{s}$$

The problem as stated has only three significant digits, so we round the final answer appropriately.

**Double check:** From the plot above, we can already graphically estimate that the maximum is somewhere around $4\frac{1}{4}$ s, which is consistent with our numerical solution to 2 significant figures. The dimensions of our answer are given in the problem, so we know that $t$ is in seconds. Since we solved $dm/dt(t)$ for $t$, the units must be the same as those given, with $t$ still in seconds – our units are correct.

**2.** Antarctica is roughly semicircular, with a radius of 2000 km. The average thickness of its ice cover is 3000 m. How many cubic centimeters of ice does Antarctica contain? (Ignore the curvature of the earth.)

[ii] You can read the symbol $\forall$ above as "for all." Thus, $\forall\ t > 0$ is read as "for all $t$ greater than zero."

**Given:** Assumption that Antarctica is a semicircular slab, dimensions of said slab and thickness of overlying ice cover.

**Find:** The number of cubic centimeters of ice. Cubic centimeters are units of *volume*, so what we are really asked for is the *volume* of ice.

**Sketch:** We assume that the ice sheet covers the entire continent, such that the ice sheet itself has a semicircular area, as shown below in Fig. ??a. As given, we neglect the curvature of the earth, and assume a flat ice sheet. We will also assume a completely uniform covering of ice, equal to the average ice cover thickness.

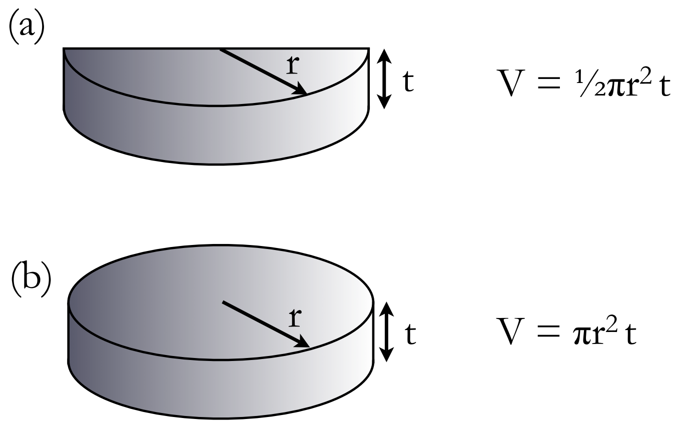


**Figure 2:** **(a)** *Proposed model for the Antarctic ice sheet: a semicircular sheet of radius r and thickness t.* **(b)** *The semicircular sheet is just half of a cylinder of radius r and thickness t. Thus, the volume of the semicircular sheet is just half the volume of the corresponding cylinder.*

**Relevant equations:**
Let our semicircular sheet have a radius $r$ and thickness $t$. As shown in Fig. ??, the semicircular sheet is just half of a cylinder of radius $r$ and thickness $t$. Thus, the volume of the semicircular sheet is just half the volume of the corresponding cylinder. The volume of a cylinder of radius $r$ and thickness $t$ is the area of the circular base ($A$) times the thickness:

$$V_{\text{cyl.}} = A_{\text{ice}}t = \pi r^2 t$$

**Symbolic solution:**
Our ice sheet has half the volume of the corresponding cylinder, thus

$$V_{\text{ice}} = \frac{1}{2}\pi r^2 t$$

**Numeric solution:** We are given a radius $r = 2000\,\text{km}$ and thickness $t = 3000\,\text{m}$. We require the volume in $\text{cm}^3$. It will be easiest (arguably) to first convert all individual dimensions to cm before inserting them into our equation.

$$r = 2000\,\text{km} = \left(2\times 10^3\,\text{km}\right)\left(\frac{10^3\,\text{m}}{1\,\text{km}}\right)\left(\frac{10^2\,\text{cm}}{1\,\text{m}}\right) = 2\times 10^8\,\left(\cancel{\text{km}}\right)\left(\frac{\cancel{\text{m}}}{\cancel{\text{km}}}\right)\left(\frac{\text{cm}}{\cancel{\text{m}}}\right) = 2\times 10^8\,\text{cm}$$

$$t = 3000\,\text{m} = \left(3\times 10^3\,\text{m}\right)\left(\frac{10^2\,\text{cm}}{1\,\text{m}}\right) = 3\times 10^5\,\left(\cancel{\text{m}}\right)\left(\frac{\text{cm}}{\cancel{\text{m}}}\right) = 3\times 10^5\,\text{cm}$$

Now we can use these values in our equation for volume:

$$V_{\text{ice}} = \frac{1}{2}\pi r^2 t \approx \frac{1}{2}(3.14)\left(2\times10^{8}\,\text{cm}\right)^2\left(3\times10^{5}\,\text{cm}\right) = 1.57\left(4\times10^{16}\right)\left(3\times10^{5}\right)\,\text{cm}^3$$

$$V_{\text{ice}} = 18.84\times10^{21}\,\text{cm}^3 = 1.884\times10^{22}\,\text{cm}^3 \xrightarrow[\text{digits}]{\text{sign.}} 2\times10^{22}\,\text{cm}^3$$

The problem as stated has only one significant digit, so we round the final answer appropriately. This is of course a very crude model, and it would be silly to claim anything more than order-of-magnitude accuracy anyway.

**Double check:** Our answer should have units of cubic centimeters, we can verify that our formula gives the correct units by dimensional analysis. Let $r$ and $t$ be given in cm. Then

$$V_{\text{ice}} = \frac{1}{2}\pi r^2 t$$

$$[V_{\text{ice}}] = [\text{cm}]^2\,[\text{cm}] = \text{cm}^3$$

As required, our formula does give the correct units for the volume of ice.

We can also simply look up the area of Antarctica: according to `http://en.wikipedia.org/wiki/Antarctic` it is about $1.4\times10^{7}\,\text{km}^2$ . Converting this to cm$^2$:

$$1.4\times10^{7}\,\text{km}^2\left(\frac{10^3\,\text{m}}{1\,\text{km}}\right)^2\left(\frac{10^2\,\text{cm}}{1\,\text{m}}\right)^2 = 1.4\times10^{7}\,\text{km}^2\left(\frac{10^6\,\text{m}^2}{1\,\text{km}^2}\right)\left(\frac{10^4\,\text{cm}^2}{1\,\text{m}^2}\right) = 1.4\times10^{17}\,\text{cm}^2$$

The volume of the ice sheet is still area times average thickness, or

$$V_{\text{ice}} = A_{\text{ice}}t_{\text{ice}} = \left(1.4\times10^{17}\,\text{cm}^2\right)\left(3\times10^{5}\,\text{cm}\right) \approx 4.2\times10^{22}\,\text{cm}^3 \xrightarrow[\text{digits}]{\text{sign.}} 4\times10^{22}\,\text{cm}^3$$

Using the actual area of Antarctica and the average ice sheet thickness, our answer is within a factor two.

Finally: we can check against a known estimate of the ice sheet volume. According to `http://en.wikipedia.org/wiki/Antarctic_ice_sheet`, the actual volume is close to $30\times10^{6}\,\text{km}^3$, or $3\times10^{22}\,\text{cm}^3$. Our estimate is within 50%; not bad for a model which is clearly only meant as a crude order-of-magnitude estimate. Modeling Antarctica as a half cylinder is perhaps not so silly.

## Problems due 15 January 2009.

**3.** A person walks in the following pattern: 3.1 km north, then 2.4 km west, and finally 5.2 km south. How far, and in what direction would a bird fly in a straight line from the same starting point to the same final point?

**Given:** Three sets of distances and directions indicating a person's walking pattern. We may regard these as three successive vectors, since both magnitude and direction are important.

**Find:** The bird would have to fly from the person's starting point to their final position, meaning we want the net displacement vector (including both magnitude and direction).

**Sketch:** It is easiest to start by choosing a coordinate system and origin. Since the person walks along the cardinal directions, the problem already implies a cartesian $x - y$ coordinate system. Let North be the $+y$ direction, East the $+x$ direction, *etc*, with the origin chosen to be the person's initial position:

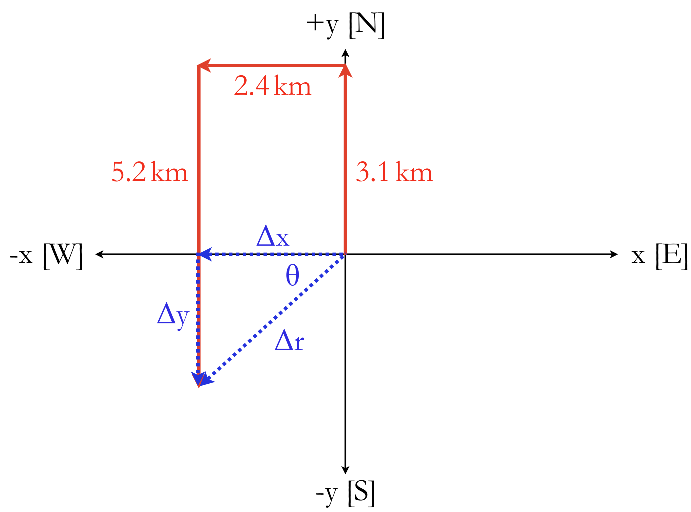


**Figure 3:** *Red solid arrows: pattern of the person walking. Blue dashed arrows: net horizontal $\Delta x$, vertical $\Delta y$, and total displacement $\Delta r$. The net displacement vector makes an angle $\theta$ with the $y$ axis.*

We can now draw in the successive walking patterns, or vectors: first 3.1 km North, in the $+y$ direction; then 2.4 km West, in the $-x$ direction; finally, 5.2 km South, in the $-y$ direction. In our chosen coordinate system, we can represent the successive walking patterns, which are individual displacements, as vectors $\vec{\mathbf{d}}_1$ through $\vec{\mathbf{d}}_3$:

$$\vec{\mathbf{d}}_1 = (3.1\,\text{km})\ \hat{\boldsymbol{\jmath}}$$
$$\vec{\mathbf{d}}_2 = (-2.4\,\text{km})\ \hat{\boldsymbol{\imath}}$$
$$\vec{\mathbf{d}}_3 = (-5.2\,\text{km})\ \hat{\boldsymbol{\jmath}}$$

**Relevant equations:** We need only know the formulas for adding vectors, finding the magnitude of a vector, and finding the angle a vector makes with the $y$ axis.

Say you have two vectors, $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. These two vectors can be written in component form as:

$$\vec{\mathbf{a}} = a_x\,\hat{\boldsymbol{\imath}} + a_y\,\hat{\boldsymbol{\jmath}}$$
$$\vec{\mathbf{b}} = b_x\,\hat{\boldsymbol{\imath}} + b_y\,\hat{\boldsymbol{\jmath}}$$

Adding vectors is done by components:

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = (a_x + b_x)\ \hat{\boldsymbol{\imath}} + (a_y + b_y)\ \hat{\boldsymbol{\jmath}}$$

The magnitude of a vector is defined as:

$$|\vec{\mathbf{a}}| = a = \sqrt{a_x^2 + a_y^2}$$

The angle $\theta$ a vector makes with the $x$ axis is given by its slope:

$$\tan\theta = \frac{a_y}{a_x}$$

**Symbolic solution:** The net displacement $\Delta\vec{\mathbf{r}}$ is just the vector sum of the individual displacements:

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{d}}_1 + \vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3$$

Once found, this net displacement can be broken down into net horizontal and vertical components $\Delta x$ and $\Delta y$, which will also give the magnitude of the displacement $\Delta r$

$$\Delta\vec{\mathbf{r}} = \Delta x\,\hat{\boldsymbol{\imath}} + \Delta y\,\hat{\boldsymbol{\jmath}}$$
$$\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

We can then find the angle $\theta$ the net displacement makes with $x$ axis:

$$\tan\theta = \frac{\Delta y}{\Delta x} \quad \text{or} \quad \theta = \tan^{-1}\frac{\Delta y}{\Delta x}$$

**Numeric solution:** Plugging in our numbers and adding by components:

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{d}}_1 + \vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3 = (-2.4\,\mathrm{km})\ \hat{\boldsymbol{\imath}} + (3.1\,\mathrm{km} - 5.2\,\mathrm{km})\ \hat{\boldsymbol{\jmath}} = (-2.4\,\mathrm{km})\ \hat{\boldsymbol{\imath}} + (-2.1\,\mathrm{km})\ \hat{\boldsymbol{\jmath}}$$

The components of the net displacement are $\Delta x = -2.4\,\mathrm{km}$ in the horizontal direction and $\Delta y = -2.1\,\mathrm{km}$ in the vertical direction. The total displacement the bird must fly is then

$$\Delta r = \sqrt{(-2.4\,\mathrm{km})^2 + (-2.1\,\mathrm{km})^2} = 3.189\,\mathrm{km} \xrightarrow[\text{digits}]{\text{sign.}} 3.2\,\mathrm{km}$$

The angle is also readily found from the horizontal and vertical displacements:

$$\theta = \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right) = \tan^{-1}\left(\frac{-2.1}{-2.4}\right) = \tan^{-1}(0.875) = 41.19° \xrightarrow[\text{digits}]{\text{sign.}} 41°$$

Referring to the cardinal directions, we may also say this is 41° south of west.

**Double check:** *Units.* The net displacement is found by adding together three vectors that have units of km, and a vector sum has the same units as the individual vectors making up the sum. The magnitude of the displacement has the same units as the displacement itself, so it is also in km.

The angle is really dimensionless, since it is the tangent of the ratio of the $x$ and $y$ displacements. Remember that an angle is really just a ratio between an arclength and a radius, we specify either degrees or radians just as an indication of how many angular units are present in one full circle ($2\pi$ radians or 360° degrees). The argument of the tan function is also required to be dimensionless, as we find above.

*Order-of-magnitude.* We can also add the vectors geometrically, which must give the same result as adding by components (as it is really the same thing). Referring to the sketch above, we want to find the length of $\Delta r$ using the blue triangle. The vertical leg $\Delta y$ must be the difference between the two vertical arrows, or 2.1 km, and the horizontal leg $\Delta x$ must be 2.4 km. The hypotenuse $\Delta r$ is then just

$$\Delta r = \sqrt{(-2.4\,\mathrm{km})^2 + (-2.1\,\mathrm{km})^2} = 3.189\,\mathrm{km} \xrightarrow[\text{digits}]{\text{sign.}} 3.2\,\mathrm{km}$$

**4.** Here are two vectors:

$$\vec{\mathbf{a}} = 4.0\,\hat{\imath} + 3.0\,\hat{\jmath} \qquad \vec{\mathbf{b}} = 6.0\,\hat{\imath} + 8.0\,\hat{\jmath}$$

Find the following quantities:

- the magnitude of $\vec{\mathbf{a}}$
- the angle of $\vec{\mathbf{a}}$ relative to $\vec{\mathbf{b}}$
- the magnitude and angle of $\vec{\mathbf{a}} + \vec{\mathbf{b}}$
- the magnitude and angle of $\vec{\mathbf{a}} - \vec{\mathbf{b}}$

**Given:** Two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ in two-dimensional cartesian coordinates.

**Find:** The magnitude of $\vec{\mathbf{a}}$, the angle of $\vec{\mathbf{a}}$ relative to $\vec{\mathbf{b}}$, and the magnitude and angle of $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ and $\vec{\mathbf{a}} - \vec{\mathbf{b}}$. For the latter, it is implied that we want the angle *with respect to the $x$ axis.*

**Sketch:** First, we define the $\hat{\imath}$ and $\hat{\jmath}$ directions, then we can draw the individual vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ as well as their sum and difference.

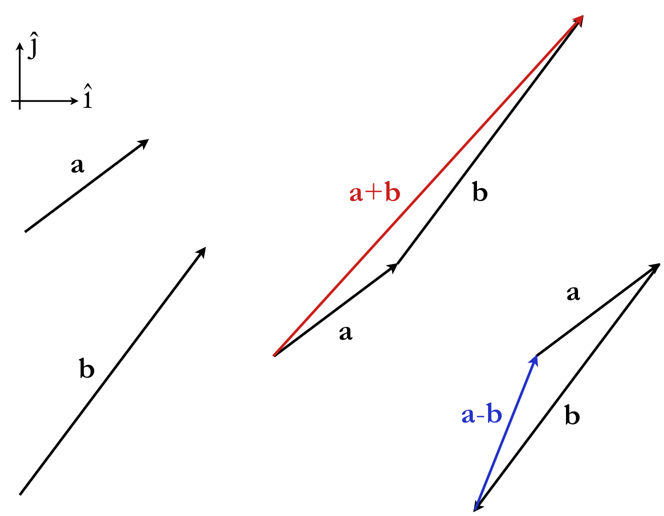


**Figure 4:** *Graphically representing the vectors* $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, *their sum* $\vec{\mathbf{a}} + \vec{\mathbf{b}}$, *and their difference* $\vec{\mathbf{a}} - \vec{\mathbf{b}}$.

**Relevant equations:** We need to know the formulas for adding vectors, finding the magnitude of a vector, and finding the angle a vector makes with the $x$ axis, which we listed in the previous problem. We also need to know that subtracting a vector is the same as adding its inverse: $\vec{\mathbf{a}}+\vec{\mathbf{b}}=\vec{\mathbf{a}}+\left(-\vec{\mathbf{b}}\right)$.

Finally, to find the angle $\varphi$ between two vectors, we can make use of the scalar product:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = ab\cos\varphi_{ab}$$

$$\implies \quad \varphi_{ab} = \cos^{-1}\left(\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{ab}\right)$$

Note that finding the angle $\varphi_{ab}$ requires knowing the magnitude of $\vec{\mathbf{b}}$.

**Symbolic solution:**

$$\begin{aligned}
|\vec{\mathbf{a}}| &= a = \sqrt{a_x^2 + a_y^2} \\
|\vec{\mathbf{b}}| &= b = \sqrt{b_x^2 + b_y^2} \\
\varphi_{ab} &= \cos^{-1}\left(\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{ab}\right) = \cos^{-1}\left(\frac{a_xb_x + a_yb_y}{ab}\right) \\
\vec{\mathbf{a}} + \vec{\mathbf{b}} &= (a_x + b_x)\,\hat{\boldsymbol{\imath}} + (a_y + b_y)\,\hat{\boldsymbol{\jmath}} \\
\theta_{\vec{\mathbf{a}}+\vec{\mathbf{b}}} &= \tan^{-1}\left(\frac{a_y + b_y}{a_x + b_x}\right) \\
\vec{\mathbf{a}} - \vec{\mathbf{b}} &= \vec{\mathbf{a}} + \left(-\vec{\mathbf{b}}\right) = (a_x - b_x)\,\hat{\boldsymbol{\imath}} + (a_y - b_y)\,\hat{\boldsymbol{\jmath}} \\
\theta_{\vec{\mathbf{a}}-\vec{\mathbf{b}}} &= \tan^{-1}\left(\frac{a_y - b_y}{a_x - b_x}\right)
\end{aligned}$$

**Numeric solution:** Now we can just plug in the numbers we have. No units are given.

$$\begin{aligned}
a &= \sqrt{4.0^2 + 3.0^2} = 5.0 \\
b &= \sqrt{6^2 + 8^2} = 10.0 \\
\varphi_{ab} &= \cos^{-1}\left(\frac{4.0\,(6.0) + 3.0\,(8.0)}{5.0\,(10.0)}\right) = \cos^{-1}\left(\frac{24}{25}\right) = 16.26^\circ \xrightarrow[\text{digits}]{\text{sign.}} 16^\circ \\
\vec{\mathbf{a}} + \vec{\mathbf{b}} &= (4.0 + 6.0)\,\hat{\boldsymbol{\imath}} + (3.0 + 8.0)\,\hat{\boldsymbol{\jmath}} = 10.0\,\hat{\boldsymbol{\imath}} + 11.0\,\hat{\boldsymbol{\jmath}} \\
\theta_{\vec{\mathbf{a}}+\vec{\mathbf{b}}} &= \tan^{-1}\left(\frac{11.0}{10.0}\right) = 47.7^\circ \xrightarrow[\text{digits}]{\text{sign.}} 48^\circ \\
\vec{\mathbf{a}} - \vec{\mathbf{b}} &= (4.0 - 6.0)\,\hat{\boldsymbol{\imath}} + (3.0 - 8.0)\,\hat{\boldsymbol{\jmath}} = -2.0\,\hat{\boldsymbol{\imath}} + -5.0\,\hat{\boldsymbol{\jmath}} \\
\theta_{\vec{\mathbf{a}}-\vec{\mathbf{b}}} &= \tan^{-1}\left(\frac{-5.0}{-2.0}\right) = 68.199^\circ \xrightarrow[\text{digits}]{\text{sign.}} 68^\circ
\end{aligned}$$

**Double check:** The easiest way to check in this case is just to use your sketch – drawn properly to scale – and graphically estimate the quantities required. (The figure above is drawn accurately.)

5. Here are three vectors:

$$\begin{aligned}
\vec{\mathbf{d}}_1 &= -3.0\,\hat{\boldsymbol{\imath}} + 3.0\,\hat{\boldsymbol{\jmath}} + 2.0\,\hat{\mathbf{k}} \\
\vec{\mathbf{d}}_2 &= -2.0\,\hat{\boldsymbol{\imath}} - 4.0\,\hat{\boldsymbol{\jmath}} + 2.0\,\hat{\mathbf{k}} \\
\vec{\mathbf{d}}_3 &= 2.0\,\hat{\boldsymbol{\imath}} + 3.0\,\hat{\boldsymbol{\jmath}} + 1.0\,\hat{\mathbf{k}}
\end{aligned}$$

What results from:

- $\vec{\mathbf{d}}_1 \cdot \left(\vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3\right)$
- $\vec{\mathbf{d}}_1 \cdot \left(\vec{\mathbf{d}}_2 \times \vec{\mathbf{d}}_3\right)$
- $\vec{\mathbf{d}}_1 \times \left(\vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3\right)$

**Given:** Three vectors $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$, and $\vec{\mathbf{d}}_3$ in two-dimensional cartesian coordinates.

**Find:** The result of various sums and scalar and vector products given above.

**Sketch:**
**Relevant equations:** In this case, we need only the requisite formulas for adding two vectors and taking the scalar and vector products of two vectors. Given two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$,

$$\begin{aligned}
\vec{\mathbf{a}} &= a_x\,\hat{\imath} + a_y\,\hat{\jmath} + a_z\,\hat{\mathbf{k}} \\
\vec{\mathbf{b}} &= b_x\,\hat{\imath} + b_y\,\hat{\jmath} + b_z\,\hat{\mathbf{k}}
\end{aligned}$$

Then
$$\begin{aligned}
\vec{\mathbf{a}} + \vec{\mathbf{b}} &= (a_x + b_x)\ \hat{\imath} + (a_y + b_y)\ \hat{\jmath} + (a_z + b_z)\ \hat{\mathbf{k}} \\
\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} &= a_x b_x + a_y b_y + a_z b_z \\
\vec{\mathbf{a}} \times \vec{\mathbf{b}} &= \det \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\,\hat{\mathbf{x}} + (a_z b_x - a_x b_z)\,\hat{\mathbf{y}} + (a_x b_y - a_y b_x)\,\hat{\mathbf{z}}
\end{aligned}$$

The only other thing we need remember is to work first inside the parenthesis. For example, for the first quantity, we need to find $\vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3$ first, and then calculate the scalar product of it with $\vec{\mathbf{d}}_1$.

**Symbolic solution:**
We can first find the results in a purely symbolic fashion by defining

$$\vec{\mathbf{d}}_1 = -3.0\,\hat{\imath} + 3.0\,\hat{\jmath} + 2.0\,\hat{\mathbf{k}} = d_{1x}\,\hat{\imath} + d_{1y}\,\hat{\jmath} + d_{1z}\,\hat{\mathbf{k}}$$

and similarly for $\vec{\mathbf{d}}_2$ and $\vec{\mathbf{d}}_3$. Finding the answer symbolically first has the advantage of being more amenable to double-checking our work later on …though it will require a bit more algebra in the intermediate steps. So it goes.

Starting with the first quantity, and working first inside the parenthesis:

$$\begin{aligned}
\vec{\mathbf{d}}_1 \cdot \left(\vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3\right) &= \vec{\mathbf{d}}_1 \cdot \Big[(d_{2x} + d_{3x})\ \hat{\imath} + (d_{2y} + d_{3y})\ \hat{\jmath} + (d_{2z} + d_{3z})\ \hat{\mathbf{k}}\Big] \\
&= \Big[d_{1x}\,\hat{\imath} + d_{1y}\,\hat{\jmath} + d_{1z}\,\hat{\mathbf{k}}\Big] \cdot \Big[(d_{2x} + d_{3x})\ \hat{\imath} + (d_{2y} + d_{3y})\ \hat{\jmath} + (d_{2z} + d_{3z})\ \hat{\mathbf{k}}\Big] \\
&= d_{1x}\,(d_{2x} + d_{3x}) + d_{1y}\,(d_{2y} + d_{3y}) + d_{1z}\,(d_{2z} + d_{3z})
\end{aligned}$$

For the second quantity, we first need to calculate the cross product of the second and third vectors. It is a bit messy, but brute force is really the only way forward.

$$
\begin{aligned}
\vec{\mathbf{d}}_1 \cdot \left(\vec{\mathbf{d}}_2 \times \vec{\mathbf{d}}_3\right) &= \vec{\mathbf{d}}_1 \cdot \left[\left(d_{2y}d_{3z} - d_{2z}d_{3y}\right)\hat{\boldsymbol{\imath}} + \left(d_{2z}d_{3x} - d_{2x}d_{3z}\right)\hat{\boldsymbol{\jmath}} + \left(d_{2x}d_{3y} - d_{2y}d_{3x}\right)\hat{\mathbf{k}}\right] \\
&= \left[d_{1x}\,\hat{\boldsymbol{\imath}} + d_{1y}\,\hat{\boldsymbol{\jmath}} + d_{1z}\,\hat{\mathbf{k}}\right] \cdot \left[\left(d_{2y}d_{3z} - d_{2z}d_{3y}\right)\hat{\boldsymbol{\imath}} + \left(d_{2z}d_{3x} - d_{2x}d_{3z}\right)\hat{\boldsymbol{\jmath}} + \left(d_{2x}d_{3y} - d_{2y}d_{3x}\right)\hat{\mathbf{k}}\right] \\
&= d_{1x}\left(d_{2y}d_{3z} - d_{2z}d_{3y}\right) + d_{1y}\left(d_{2z}d_{3x} - d_{2x}d_{3z}\right) + d_{1z}\left(d_{2x}d_{3y} - d_{2y}d_{3x}\right)
\end{aligned}
$$

The third quantity is no more difficult; this time we first perform the addition, and then take a cross product:

$$
\begin{aligned}
\vec{\mathbf{d}}_1 \times \left(\vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3\right) &= \vec{\mathbf{d}}_1 \times \left[\left(d_{2x} + d_{3x}\right)\hat{\boldsymbol{\imath}} + \left(d_{2y} + d_{3y}\right)\hat{\boldsymbol{\jmath}} + \left(d_{2z} + d_{3z}\right)\hat{\mathbf{k}}\right] \\
&= \left[d_{1y}\left(d_{2z} + d_{3z}\right) - d_{1z}\left(d_{2y} + d_{3y}\right)\right]\hat{\boldsymbol{\imath}} + \left[d_{1z}\left(d_{2x} + d_{3x}\right) - d_{1x}\left(d_{2z} + d_{3z}\right)\right]\hat{\boldsymbol{\jmath}} \\
&\qquad + \left[d_{1x}\left(d_{2y} + d_{3y}\right) - d_{1y}\left(d_{2x} + d_{3x}\right)\right]\hat{\mathbf{k}}
\end{aligned}
$$

There is not much point in simplifying further, there are no like terms to collect.

**Numeric solution:**
All that is needed now is to plug in the actual numbers, noting that $d_{1x} = -3.0$, $d_{1y} = 3.0$, $d_{1z} = 2.0$, *etc.* For the first quantity:

$$
\begin{aligned}
\vec{\mathbf{d}}_1 \cdot \left(\vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3\right) &= d_{1x}\left(d_{2x} + d_{3x}\right) + d_{1y}\left(d_{2y} + d_{3y}\right) + d_{1z}\left(d_{2z} + d_{3z}\right) \\
&= -3.0\left(-2.0 + 2.0\right) + 3.0\left(-4.0 + 3.0\right) + 2.0\left(2.0 + 1.0\right) = 0 - 3.0 + 6.0 = 3.0
\end{aligned}
$$

For the second quantity:

$$
\begin{aligned}
\vec{\mathbf{d}}_1 \cdot \left(\vec{\mathbf{d}}_2 \times \vec{\mathbf{d}}_3\right) &= d_{1x}\left(d_{2y}d_{3z} - d_{2z}d_{3y}\right) + d_{1y}\left(d_{2z}d_{3x} - d_{2x}d_{3z}\right) + d_{1z}\left(d_{2x}d_{3y} - d_{2y}d_{3x}\right) \\
&= -3.0\left(-4.0 - 6.0\right) + 3.0\left(4.0 + 2.0\right) + 2.0\left(-6.0 + 8.0\right) = 30 + 18.0 + 4.0 = 52.0
\end{aligned}
$$

For the third quantity:

$$
\begin{aligned}
&\left[d_{1y}\left(d_{2z} + d_{3z}\right) - d_{1z}\left(d_{2y} + d_{3y}\right)\right]\hat{\boldsymbol{\imath}} + \left[d_{1z}\left(d_{2x} + d_{3x}\right) - d_{1x}\left(d_{2z} + d_{3z}\right)\right]\hat{\boldsymbol{\jmath}} \\
&\qquad + \left[d_{1x}\left(d_{2y} + d_{3y}\right) - d_{1y}\left(d_{2x} + d_{3x}\right)\right]\hat{\mathbf{k}} \\
&= \left[3.0\left(2.0 + 1.0\right) - 2.0\left(-4.0 + 3.0\right)\right]\hat{\boldsymbol{\imath}} + \left[2.0\left(-2.0 + 2.0\right) + 3.0\left(2.0 + 1.0\right)\right]\hat{\boldsymbol{\jmath}} \\
&\qquad + \left[-3.0\left(-4.0 + 3.0\right) - 3.0\left(-2.0 + 2.0\right)\right]\hat{\mathbf{k}} \\
&= \left[9.0 + 2.0\right]\hat{\boldsymbol{\imath}} + \left[0 + 9.0\right]\hat{\boldsymbol{\jmath}} + \left[3.0 + 0\right]\hat{\mathbf{k}} \\
&= 11.0\,\hat{\boldsymbol{\imath}} + 9.0\,\hat{\boldsymbol{\jmath}} + 3.0\,\hat{\mathbf{k}}
\end{aligned}
$$

**Double check:** *Units. Order-of-magnitude.*
There are no units in this problem, but we can decide what sort of solution should we expect *qualitatively* – should the answers be vectors, scalars, or neither?

For the first quantity, the quantity inside parenthesis is the sum of two vectors, and therefore a vector itself. We then need to find the scalar product of this vector with $\vec{\mathbf{d}}_1$. The final quantity, then, is the scalar product of two vectors, which is itself a scalar (*i.e.,* just a number). This also means that the product should have only terms with the product of two components, such as $d_{1x}d_{2x}$, which is consistent with our answer.

The second quantity is similarly a scalar, since the cross product in parenthesis results in an (axial) vector, whose scalar product with $\vec{\mathbf{d}}_1$ also gives a scalar. Since there are two products here, the final answer should have only terms with three components, such as $d_{1x}d_{2y}d_{3z}$, consistent with our answer.

The third quantity has a vector resulting in the parenthesis, and we need its vector product with $\vec{\mathbf{d}}_1$, which gives an (axial) vector. Thus, only the third quantity is a vector at all, and only a pseudovector at that, the other two are just numbers. Again, we have only one product here, so the final answer should again have terms with two components.

## Problems due 16 January 2009.

**6.** A hoodlum throws a stone vertically downward with an initial speed of 12.0 m/s from the roof of a building 30.0 m above the ground. How long does it take the stone to reach the ground, and what is its speed on impact?

**Given:** the initial velocity and position of a stone undergoing free-fall motion.

**Find:** The time required for the stone to reach the ground, 30 m below its starting point, and its speed on impact. The stone is thrown straight down, after which it is only under the influence of gravity. Thus, the motion will be along a straight (vertical) line, with constant acceleration.

**Sketch:** The situation is quite simple, but we can take this as an opportunity to choose a coordinate system and origin. Since the motion is one dimensional, we need only one axis. Let that be an $x$ axis, running vertically, with the $+x$ direction being upward. We will choose the origin to be the ground level, which makes the stone's initial position $x_i = 30.0$ m. With these choices, the stone's initial velocity $v_i$ is in the $-x$ direction, as is the acceleration due to gravity. For completion, let $t = 0$ be the time at which the stone is thrown.

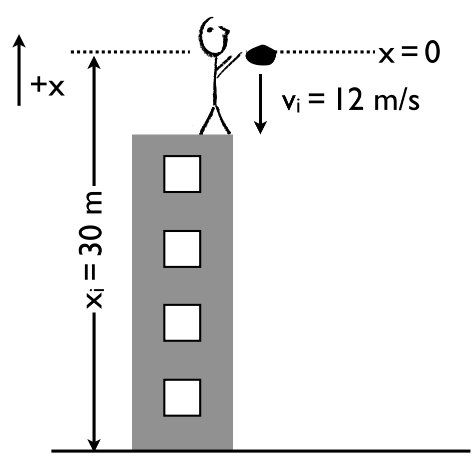


**Figure 5:** *Hoodlum throwing a stone off the roof of a tall building.*

**Relevant equations:** We derived a general equation for one-dimensional motion with constant acceleration, this is all we need along with the initial conditions. Let $x(t)$ be the stone's position at a time

$t$, $x_i$ its initial position, $v_i$ its initial velocity, and $a$ the acceleration of the stone. Then

$$x(t) = x_i + v_i t + \frac{1}{2} a t^2 \tag{1}$$

Additionally, we can get the stone's velocity (speed) by differentiating with respect to time:

$$v(t) = \frac{dx}{dt} = v_i + at \tag{2}$$

**Symbolic solution:** With our choice of origin, we want to find the time at which the stone reaches position $x\!=\!0$. From Eq. **??**:

$$x(t) = 0 = x_i + v_i t + \frac{1}{2} a t^2 \tag{3}$$

This is just a parabolic equation in $t$, for which the solution is well-known:

$$t_{x=0} = \frac{-v_i \pm \sqrt{v_i^2 - 2ax_i}}{a} \tag{4}$$

We also want to find the velocity at this time, which means plugging $t_{x=0}$ into Eq. **??**.

**Numeric solution:** With our choice of origin, and with $+x$ being in the vertical direction, we have the following boundary conditions:

$$\begin{aligned} x_i &= 30.0\,\mathrm{m} \\ v_i &= -12.0\,\mathrm{m/s} \\ a &= -g = -9.81\,\mathrm{m/s^2} \end{aligned}$$

Using these values in our solution above, the time required for the stone to reach the ground is

$$\begin{aligned} t_{x=0} &= \frac{-v_i \pm \sqrt{v_i^2 - 2ax_i}}{a} = \frac{12.0\,\mathrm{m/s} \pm \sqrt{12.0^2\,\mathrm{m^2/s^2} + 2\,(9.81\,\mathrm{m/s^2})\,(30.0\,\mathrm{m})}}{-9.81\,\mathrm{m/s^2}} \\ &\approx \{1.54\,\mathrm{s}, -3.99\,\mathrm{s}\} \end{aligned}$$

We reject the negative root as unphysical – this solution is a time before the ball was thrown, which makes no sense. This solution is *mathematically* valid, but our problem is physically meaningful only for $t > 0$. Taking the positive root as our solution, the ball hits the ground about $1.54\,\mathrm{s}$ after it is thrown.

At the point the ball hits the ground, the velocity is

$$v(1.54\,\mathrm{s}) = v_i + at \approx -12.0\,\mathrm{m/s} - \left(9.8\,\mathrm{m/s^2}\right)(1.54\,\mathrm{s}) = -27.1\,\mathrm{m/s}$$

**Double check:** If we ignore the acceleration due to gravity, at an initial velocity of $12.0\,\mathrm{m/s}$, the stone should cover $30.0\,\mathrm{m}$ in $2.5\,\mathrm{s}$ ($x\!=\!v_i t$). On the other hand, if we ignore the initial velocity, and consider only free-fall motion, it should take about $1.75\,\mathrm{s}$ ($x = \frac{1}{2} g t^2$). The real answer should be slightly less time than either of these, since we are combining the effects of gravity and the initial velocity, both of which act in the same direction. Our answer of $1.5\,\mathrm{s}$ seems reasonable in this light.

We can also check the units of Eq. **??** with dimensional analysis to be sure:

$$t_{x=0} = \frac{[\mathrm{m/s}] \pm \sqrt{[\mathrm{m/s}]^2 - 2[\mathrm{m/s^2}][\mathrm{m}]}}{\mathrm{m/s^2}} = \frac{[\mathrm{m/s}] \pm \sqrt{[\mathrm{m^2/s^2}]}}{\mathrm{m/s^2}} = \frac{[\mathrm{m/s}] \pm [\mathrm{m/s}]}{\mathrm{m/s^2}} = [\mathrm{s}]$$

The units are seconds, as we require.

**7.** The position of a particle moving along the $x$ axis is given in centimeters by

$$x = 9.75 + 1.50t^3$$

where $t$ is in seconds. Calculate the instantaneous velocity and acceleration at $t = 2.50\,\mathrm{s}$.

**Given:** position versus time $x(t)$ in meters and seconds.

**Find:** Instantaneous velocity and acceleration at $t = 2.50\,\mathrm{s}$.

**Sketch:** It is somewhat helpful to graph the position versus time. Most graphing programs these days will also calculate derivatives for you, so it is little extra work to plot velocity and acceleration as well.

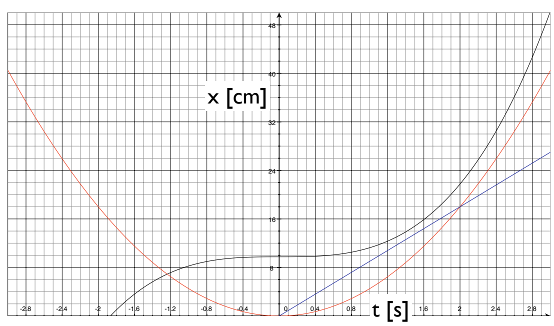


**Figure 6:** *Position (black), velocity (red), and acceleration (blue) as a function of time.*

**Relevant equations:** We need only the definitions of instantaneous velocity and acceleration:

$$v(t) = \frac{d}{dt}x(t) \tag{5}$$
$$a(t) = \frac{d}{dt}v(t) = \frac{d^2}{dt^2}x(t) \tag{6}$$

**Symbolic solution:**

$$v(t) = \frac{d}{dt}\left(9.75 + 1.50t^3\right) = 4.50t^2 \tag{7}$$
$$a(t) = \frac{d}{dt}\left(4.50t^2\right) = 9t \tag{8}$$

**Numeric solution:** Evaluating the equations above at $t = 2.50\,\text{s}$,

$$v(2.50\,\text{s}) \approx 28.1\,\text{cm/s}$$
$$a(2.50\,\text{s}) \approx 22.5\,\text{cm/s}^2$$

**Double check:** We can simply read the values off of the graph above, which gives consistent results.

Dimensional analysis on the $x(t)$ equation given will tell us what the units of the constants must be, which will let us check our answer. All terms must have units of meters. Thus, the constant 9.75 must have units of cm, while the constant 1.5 must have units cm/s$^3$, since it is multiplied by $t^3$ in s$^3$ and results in m. Equation **??** has then a constant in cm/s$^3$ multiplied by $t^2$ in s$^2$, giving cm/s, while Eq. **??** has a constant in cm/s$^3$ multiplied by $t$ in s, giving cm/s$^2$. Both have the correct units.

**8.** Two trains are 100 km apart on the same track, headed on a collision course towards each other. Both are traveling 50 km per hour. A very speedy bird takes off from the first train and flies at 75 km per hour toward the second train. The bird then immediately turns around and flies back to the first train. Then he flies back to the second train, and repeats the process over and over as the distance between the trains diminishes. How far will he have flown before the trains collide?

This is a rather famous problem, which has a very simple solution. The two trains are going toward each other at 50 km/h, meaning they close the distance between them at a rate of 100 km/h. Thus, they will meet in precisely 1 hr. The bird flies then for a total of one hour at 75 km/h, and hence it must cover a distance of 75 km.

There is also a more painful infinite series solution to this problem. See, for example,

`http://scienceblogs.com/builtonfacts/2008/12/bouncing_birds.php`

for some background and discussion on this problem.