UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 125 / LeClair Spring 2009

Problem Set 1: Solutions 1

Problems due 13 January 2009

1. Water is poured into a container that has a leak. The mass m of the water is as a function of time t is

$$m = 5.00t^{0.8} - 3.00t + 20.00$$

with $t \geq 0$, m in grams, and t in seconds. At what time is the water mass greatest?

Given: Water mass versus time $m(t)$.

Find: The time t at which the water mass m is greatest. This can be accomplished by finding the time derivative of $m(t)$ and setting it equal to zero, followed by checking the second derivative to be sure we have found a maximum.

Sketch: It is useful to plot the function $m(t)$ and graphically estimate about where the maximum should be, roughly.[i]

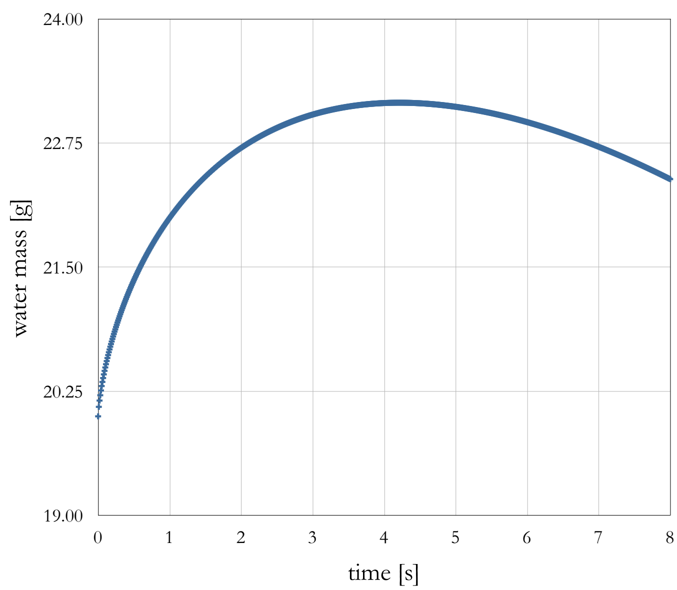

Figure 1: *Water mass versus time, problem 1. Note the rather expanded vertical axis, with offset origin.*

It is clear that there is indeed a maximum water mass, and it occurs just after $t = 4$ s.

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[i]It is relatively easy to do this on a graphing calculator, which can be found online these days: `http://www.coolmath.com/graphit/`.

Relevant equations: We need to find the maximum of $m(t)$. Therefore, we need to set the first derivative equal to zero. We must also check that the second derivative is negative to ensure that we have found a maximum, not a minimum. Therefore, only two equations are needed:

$$\frac{dm}{dt} = \frac{d}{dt}\left[m(t)\right] = 0 \qquad \text{and} \qquad \frac{d^2m}{dt^2} = \frac{d^2}{dt^2}\left[m(t)\right] < 0 \quad \Longrightarrow \quad \text{maximum in } m(t)$$

Symbolic solution:

$$\begin{aligned}
\frac{dm}{dt} &= \frac{d}{dt}\left[5t^{0.8} - 3t + 20\right] = 0.8\left(5t^{0.8-1}\right) - 3 = 4t^{-0.2} - 3 = 0 \\
4t^{-0.2} - 3 &= 0 \\
t^{-0.2} &= \frac{3}{4} \\
\Longrightarrow \quad t &= \left(\frac{3}{4}\right)^{-5} = \left(\frac{4}{3}\right)^{5}
\end{aligned}$$

Thus, $m(t)$ takes on an extreme value at $t = (4/3)^5$. We did not prove whether it is a maximum or a minimum however! This is important … so we should apply the *second derivative* test.

Recall briefly that after finding the extreme point of a function $f(x)$ via $df/dx|_{x=a} = 0$, one should calculate $d^2f/dx^2|_{x=a}$: if $d^2f/dx^2|_{x=a} < 0$, you have a maximum, if $d^2f/dx^2|_{x=a} > 0$ you have a minimum, and if $d^2f/dx^2|_{x=a} = 0$, the test basically wasted your time. Anyway:

$$\begin{aligned}
\frac{d^2m}{dt^2} &= \frac{d}{dt}\left[\frac{dm}{dt}\right] = \frac{d}{dt}\left[4t^{-0.2} - 3\right] = -0.2\left(4t^{-0.2-1}\right) = -0.8t^{-1.2} \\
\frac{d^2m}{dt^2} &< 0 \quad \forall \quad t > 0
\end{aligned}$$

$\frac{d^2m}{dt^2}$ is greater than zero[ii] $t > 0$, since $t^{-1.2}$ is always positive in that regime, which means we have indeed found a maximum.

Numeric solution: Evaluating our answer numerically, remembering that t has units of seconds (s):

$$t = \left(\frac{4}{3}\right)^5 \approx 4.21399 \xrightarrow[\text{digits}]{\text{sign.}} 4.21\,\text{s}$$

The problem as stated has only three significant digits, so we round the final answer appropriately.

Double check: From the plot above, we can already graphically estimate that the maximum is somewhere around $4\frac{1}{4}$ s, which is consistent with our numerical solution to 2 significant figures. The dimensions of our answer are given in the problem, so we know that t is in seconds. Since we solved $dm/dt(t)$ for t, the units must be the same as those given, with t still in seconds – our units are correct.

2. Antarctica is roughly semicircular, with a radius of 2000 km. The average thickness of its ice cover is 3000 m. How many cubic centimeters of ice does Antarctica contain? (Ignore the curvature of the earth.)

[ii] You can read the symbol $\forall$ above as "for all." Thus, $\forall\ t > 0$ is read as "for all t greater than zero."

Given: Assumption that Antarctica is a semicircular slab, dimensions of said slab and thickness of overlying ice cover.

Find: The number of cubic centimeters of ice. Cubic centimeters are units of *volume*, so what we are really asked for is the *volume* of ice.

Sketch: We assume that the ice sheet covers the entire continent, such that the ice sheet itself has a semicircular area, as shown below in Fig. 2a. As given, we neglect the curvature of the earth, and assume a flat ice sheet. We will also assume a completely uniform covering of ice, equal to the average ice cover thickness.

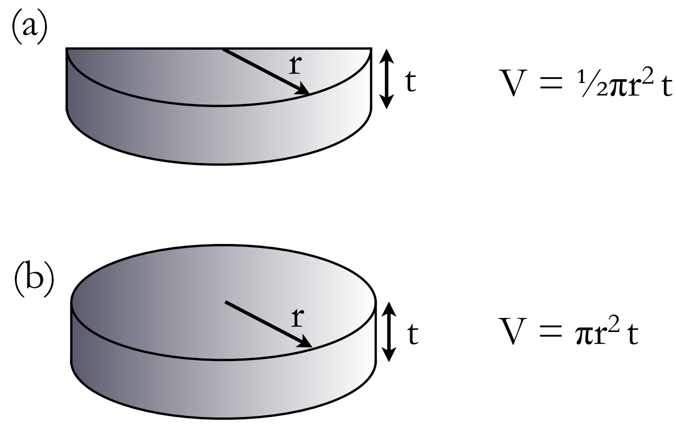

Figure 2: **(a)** *Proposed model for the Antarctic ice sheet: a semicircular sheet of radius r and thickness t.* **(b)** *The semicircular sheet is just half of a cylinder of radius r and thickness t. Thus, the volume of the semicircular sheet is just half the volume of the corresponding cylinder.*

Relevant equations:
Let our semicircular sheet have a radius r and thickness t. As shown in Fig. 2, the semicircular sheet is just half of a cylinder of radius r and thickness t. Thus, the volume of the semicircular sheet is just half the volume of the corresponding cylinder. The volume of a cylinder of radius r and thickness t is the area of the circular base (A) times the thickness:

$$V_{\text{cyl.}} = A_{\text{ice}} t = \pi r^2 t$$

Symbolic solution:
Our ice sheet has half the volume of the corresponding cylinder, thus

$$V_{\text{ice}} = \frac{1}{2}\pi r^2 t$$

Numeric solution: We are given a radius $r = 2000\,\text{km}$ and thickness $t = 3000\,\text{m}$. We require the volume in cm^3. It will be easiest (arguably) to first convert all individual dimensions to cm before inserting them into our equation.

$$r = 2000\,\text{km} = \left(2 \times 10^3\,\text{km}\right)\left(\frac{10^3\,\text{m}}{1\,\text{km}}\right)\left(\frac{10^2\,\text{cm}}{1\,\text{m}}\right) = 2 \times 10^8\,\left(\cancel{\text{km}}\right)\left(\frac{\cancel{\text{m}}}{\cancel{\text{km}}}\right)\left(\frac{\text{cm}}{\cancel{\text{m}}}\right) = 2 \times 10^8\,\text{cm}$$

$$t = 3000\,\text{m} = \left(3 \times 10^3\,\text{m}\right)\left(\frac{10^2\,\text{cm}}{1\,\text{m}}\right) = 3 \times 10^5\,\left(\cancel{\text{m}}\right)\left(\frac{\text{cm}}{\cancel{\text{m}}}\right) = 3 \times 10^5\,\text{cm}$$

Now we can use these values in our equation for volume:

$$V_{\text{ice}} = \frac{1}{2}\pi r^2 t \approx \frac{1}{2}(3.14)\left(2\times10^8\,\text{cm}\right)^2\left(3\times10^5\,\text{cm}\right) = 1.57\left(4\times10^{16}\right)\left(3\times10^5\right)\,\text{cm}^3$$
$$V_{\text{ice}} = 18.84\times10^{21}\,\text{cm}^3 = 1.884\times10^{22}\,\text{cm}^3 \xrightarrow[\text{digits}]{\text{sign.}} 2\times10^{22}\,\text{cm}^3$$

The problem as stated has only one significant digit, so we round the final answer appropriately. This is of course a very crude model, and it would be silly to claim anything more than order-of-magnitude accuracy anyway.

Double check: Our answer should have units of cubic centimeters, we can verify that our formula gives the correct units by dimensional analysis. Let r and t be given in cm. Then

$$V_{\text{ice}} = \frac{1}{2}\pi r^2 t$$
$$[V_{\text{ice}}] = [\text{cm}]^2\,[\text{cm}] = \text{cm}^3$$

As required, our formula does give the correct units for the volume of ice.

We can also simply look up the area of Antarctica: according to `http://en.wikipedia.org/wiki/Antarctic` it is about $1.4\times10^7\,\text{km}^2$. Converting this to cm^2:

$$1.4\times10^7\,\text{km}^2\left(\frac{10^3\,\text{m}}{1\,\text{km}}\right)^2\left(\frac{10^2\,\text{cm}}{1\,\text{m}}\right)^2 = 1.4\times10^7\,\text{km}^2\left(\frac{10^6\,\text{m}^2}{1\,\text{km}^2}\right)\left(\frac{10^4\,\text{cm}^2}{1\,\text{m}^2}\right) = 1.4\times10^{17}\,\text{cm}^2$$

The volume of the ice sheet is still area times average thickness, or

$$V_{\text{ice}} = A_{\text{ice}}t_{\text{ice}} = \left(1.4\times10^{17}\,\text{cm}^2\right)\left(3\times10^5\,\text{cm}\right) \approx 4.2\times10^{22}\,\text{cm}^3 \xrightarrow[\text{digits}]{\text{sign.}} 4\times10^{22}\,\text{cm}^3$$

Using the actual area of Antarctica and the average ice sheet thickness, our answer is within a factor two.

Finally: we can check against a known estimate of the ice sheet volume. According to `http://en.wikipedia.org/wiki/Antarctic_ice_sheet`, the actual volume is close to $30\times10^6\,\text{km}^3$, or $3\times10^{22}\,\text{cm}^3$. Our estimate is within 50%; not bad for a model which is clearly only meant as a crude order-of-magnitude estimate. Modeling Antarctica as a half cylinder is perhaps not so silly.