UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 125 / LeClair — Spring 2009

Problem Set 1: Solutions 2

Problems due 15 January 2009.

1. A person walks in the following pattern: 3.1 km north, then 2.4 km west, and finally 5.2 km south. How far, and in what direction would a bird fly in a straight line from the same starting point to the same final point?

Given: Three sets of distances and directions indicating a person's walking pattern. We may regard these as three successive vectors, since both magnitude and direction are important.

Find: The bird would have to fly from the person's starting point to their final position, meaning we want the net displacement vector (including both magnitude and direction).

Sketch: It is easiest to start by choosing a coordinate system and origin. Since the person walks along the cardinal directions, the problem already implies a cartesian $x-y$ coordinate system. Let North be the $+y$ direction, East the $+x$ direction, *etc*, with the origin chosen to be the person's initial position:

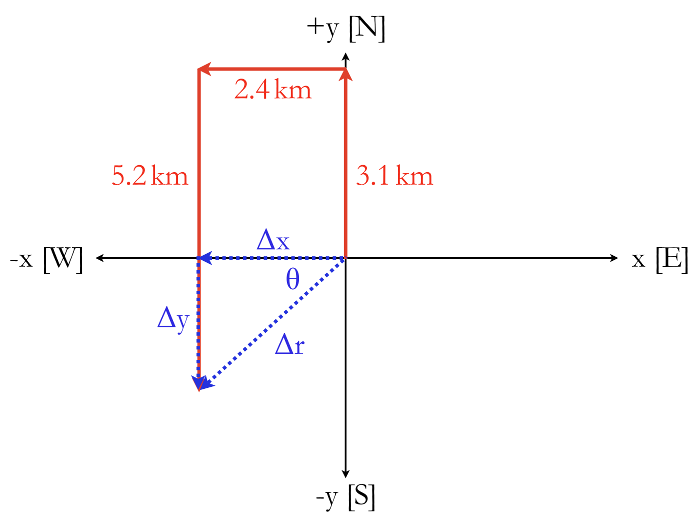

Figure 1: *Red solid arrows: pattern of the person walking. Blue dashed arrows: net horizontal Δx, vertical Δy, and total displacement Δr. The net displacement vector makes an angle θ with the y axis.*

We can now draw in the successive walking patterns, or vectors: first 3.1 km North, in the $+y$ direction; then 2.4 km West, in the $-x$ direction; finally, 5.2 km South, in the $-y$ direction. In our chosen coordinate system, we can represent the successive walking patterns, which are individual displacements, as vectors $\vec{\mathbf{d}}_1$ through $\vec{\mathbf{d}}_3$:

$$\vec{\mathbf{d}}_1 = (3.1\,\text{km})\;\hat{\boldsymbol{\jmath}}$$
$$\vec{\mathbf{d}}_2 = (-2.4\,\text{km})\;\hat{\boldsymbol{\imath}}$$
$$\vec{\mathbf{d}}_3 = (-5.2\,\text{km})\;\hat{\boldsymbol{\jmath}}$$

Relevant equations: We need only know the formulas for adding vectors, finding the magnitude of a vector, and finding the angle a vector makes with the y axis.

Say you have two vectors, $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. These two vectors can be written in component form as:

$$\vec{\mathbf{a}} = a_x\,\hat{\boldsymbol{\imath}} + a_y\,\hat{\boldsymbol{\jmath}}$$
$$\vec{\mathbf{b}} = b_x\,\hat{\boldsymbol{\imath}} + b_y\,\hat{\boldsymbol{\jmath}}$$

Adding vectors is done by components:

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = (a_x + b_x)\;\hat{\boldsymbol{\imath}} + (a_y + b_y)\;\hat{\boldsymbol{\jmath}}$$

The magnitude of a vector is defined as:

$$|\vec{\mathbf{a}}| = a = \sqrt{a_x^2 + a_y^2}$$

The angle θ a vector makes with the x axis is given by its slope:

$$\tan\theta = \frac{a_y}{a_x}$$

Symbolic solution: The net displacement $\Delta\vec{\mathbf{r}}$ is just the vector sum of the individual displacements:

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{d}}_1 + \vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3$$

Once found, this net displacement can be broken down into net horizontal and vertical components Δx and Δy, which will also give the magnitude of the displacement Δr

$$\Delta\vec{\mathbf{r}} = \Delta x\,\hat{\boldsymbol{\imath}} + \Delta y\,\hat{\boldsymbol{\jmath}}$$
$$\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

We can then find the angle θ the net displacement makes with x axis:

$$\tan\theta = \frac{\Delta y}{\Delta x} \quad \text{or} \quad \theta = \tan^{-1}\frac{\Delta y}{\Delta x}$$

Numeric solution: Plugging in our numbers and adding by components:

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{d}}_1 + \vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3 = (-2.4\,\text{km})\;\hat{\boldsymbol{\imath}} + (3.1\,\text{km} - 5.2\,\text{km})\;\hat{\boldsymbol{\jmath}} = (-2.4\,\text{km})\;\hat{\boldsymbol{\imath}} + (-2.1\,\text{km})\;\hat{\boldsymbol{\jmath}}$$

The components of the net displacement are $\Delta x = -2.4\,\text{km}$ in the horizontal direction and $\Delta y = -2.1\,\text{km}$ in the vertical direction. The total displacement the bird must fly is then

$$\Delta r=\sqrt{(-2.4\,\mathrm{km})^2+(-2.1\,\mathrm{km})^2}=3.189\,\mathrm{km}\xrightarrow[\text{digits}]{\text{sign.}}3.2\,\mathrm{km}$$

The angle is also readily found from the horizontal and vertical displacements:

$$\theta=\tan^{-1}\left(\frac{\Delta y}{\Delta x}\right)=\tan^{-1}\left(\frac{-2.1}{-2.4}\right)=\tan^{-1}(0.875)=41.19^\circ\xrightarrow[\text{digits}]{\text{sign.}}41^\circ$$

Referring to the cardinal directions, we may also say this is 41° south of west.

Double check: *Units.* The net displacement is found by adding together three vectors that have units of km, and a vector sum has the same units as the individual vectors making up the sum. The magnitude of the displacement has the same units as the displacement itself, so it is also in km.

The angle is really dimensionless, since it is the tangent of the ratio of the x and y displacements. Remember that an angle is really just a ratio between an arclength and a radius, we specify either degrees or radians just as an indication of how many angular units are present in one full circle (2π radians or 360° degrees). The argument of the $\tan$ function is also required to be dimensionless, as we find above.

Order-of-magnitude. We can also add the vectors geometrically, which must give the same result as adding by components (as it is really the same thing). Referring to the sketch above, we want to find the length of Δr using the blue triangle. The vertical leg Δy must be the difference between the two vertical arrows, or 2.1 km, and the horizontal leg Δx must be 2.4 km. The hypotenuse Δr is then just

$$\Delta r=\sqrt{(-2.4\,\mathrm{km})^2+(-2.1\,\mathrm{km})^2}=3.189\,\mathrm{km}\xrightarrow[\text{digits}]{\text{sign.}}3.2\,\mathrm{km}$$

2. Here are two vectors:

$$\vec{\mathbf{a}}=4.0\,\hat{\imath}+3.0\,\hat{\jmath}\qquad\vec{\mathbf{b}}=6.0\,\hat{\imath}+8.0\,\hat{\jmath}$$

Find the following quantities:

- the magnitude of $\vec{\mathbf{a}}$
- the angle of $\vec{\mathbf{a}}$ relative to $\vec{\mathbf{b}}$
- the magnitude and angle of $\vec{\mathbf{a}}+\vec{\mathbf{b}}$
- the magnitude and angle of $\vec{\mathbf{a}}-\vec{\mathbf{b}}$

Given: Two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ in two-dimensional cartesian coordinates.

Find: The magnitude of $\vec{\mathbf{a}}$, the angle of $\vec{\mathbf{a}}$ relative to $\vec{\mathbf{b}}$, and the magnitude and angle of $\vec{\mathbf{a}}+\vec{\mathbf{b}}$ and $\vec{\mathbf{a}}-\vec{\mathbf{b}}$. For the latter, it is implied that we want the angle *with respect to the x axis.*

Sketch: First, we define the $\hat{\imath}$ and $\hat{\jmath}$ directions, then we can draw the individual vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ as well as their sum and difference.

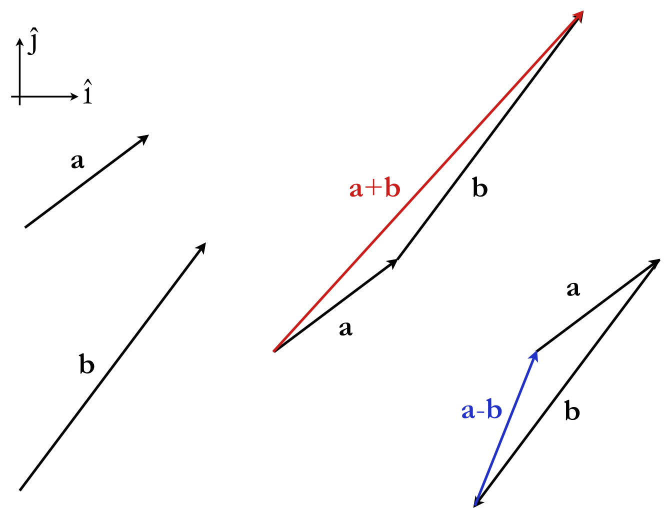

Figure 2: *Graphically representing the vectors* $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, *their sum* $\vec{\mathbf{a}}+\vec{\mathbf{b}}$, *and their difference* $\vec{\mathbf{a}}-\vec{\mathbf{b}}$.

Relevant equations: We need to know the formulas for adding vectors, finding the magnitude of a vector, and finding the angle a vector makes with the x axis, which we listed in the previous problem. We also need to know that subtracting a vector is the same as adding its inverse: $\vec{\mathbf{a}}+\vec{\mathbf{b}}=\vec{\mathbf{a}}+\left(-\vec{\mathbf{b}}\right)$.

Finally, to find the angle φ between two vectors, we can make use of the scalar product:

$$\begin{aligned} \vec{\mathbf{a}}\cdot\vec{\mathbf{b}} &= ab\cos\varphi_{ab} \\ \Longrightarrow \quad \varphi_{ab} &= \cos^{-1}\left(\frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{ab}\right) \end{aligned}$$

Note that finding the angle φ_{ab} requires knowing the magnitude of $\vec{\mathbf{b}}$.

Symbolic solution:

$$\begin{aligned} |\vec{\mathbf{a}}| &= a = \sqrt{a_x^2+a_y^2} \\ |\vec{\mathbf{b}}| &= b = \sqrt{b_x^2+b_y^2} \\ \varphi_{ab} &= \cos^{-1}\left(\frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{ab}\right) = \cos^{-1}\left(\frac{a_xb_x+a_yb_y}{ab}\right) \\ \vec{\mathbf{a}}+\vec{\mathbf{b}} &= (a_x+b_x)\,\hat{\boldsymbol{\imath}} + (a_y+b_y)\,\hat{\boldsymbol{\jmath}} \\ \theta_{\vec{\mathbf{a}}+\vec{\mathbf{b}}} &= \tan^{-1}\left(\frac{a_y+b_y}{a_x+b_x}\right) \\ \vec{\mathbf{a}}-\vec{\mathbf{b}} &= \vec{\mathbf{a}}+\left(-\vec{\mathbf{b}}\right) = (a_x-b_x)\,\hat{\boldsymbol{\imath}} + (a_y-b_y)\,\hat{\boldsymbol{\jmath}} \\ \theta_{\vec{\mathbf{a}}-\vec{\mathbf{b}}} &= \tan^{-1}\left(\frac{a_y-b_y}{a_x-b_x}\right) \end{aligned}$$

Numeric solution: Now we can just plug in the numbers we have. No units are given.

$$
\begin{aligned}
a &= \sqrt{4.0^2 + 3.0^2} = 5.0 \\
b &= \sqrt{6^2 + 8^2} = 10.0 \\
\varphi_{ab} &= \cos^{-1}\left(\frac{4.0\,(6.0) + 3.0\,(8.0)}{5.0\,(10.0)}\right) = \cos^{-1}\left(\frac{24}{25}\right) = 16.26^\circ \xrightarrow[\text{digits}]{\text{sign.}} 16^\circ \\
\vec{\mathbf{a}} + \vec{\mathbf{b}} &= (4.0 + 6.0)\ \hat{\imath} + (3.0 + 8.0)\ \hat{\jmath} = 10.0\,\hat{\imath} + 11.0\,\hat{\jmath} \\
\theta_{\vec{\mathbf{a}}+\vec{\mathbf{b}}} &= \tan^{-1}\left(\frac{11.0}{10.0}\right) = 47.7^\circ \xrightarrow[\text{digits}]{\text{sign.}} 48^\circ \\
\vec{\mathbf{a}} - \vec{\mathbf{b}} &= (4.0 - 6.0)\ \hat{\imath} + (3.0 - 8.0)\ \hat{\jmath} = -2.0\,\hat{\imath} + -5.0\,\hat{\jmath} \\
\theta_{\vec{\mathbf{a}}-\vec{\mathbf{b}}} &= \tan^{-1}\left(\frac{-5.0}{-2.0}\right) = 68.199^\circ \xrightarrow[\text{digits}]{\text{sign.}} 68^\circ
\end{aligned}
$$

Double check: The easiest way to check in this case is just to use your sketch – drawn properly to scale – and graphically estimate the quantities required. (The figure above is drawn accurately.)

3. Here are three vectors:

$$
\begin{aligned}
\vec{\mathbf{d}}_1 &= -3.0\,\hat{\imath} + 3.0\,\hat{\jmath} + 2.0\,\hat{\mathbf{k}} \\
\vec{\mathbf{d}}_2 &= -2.0\,\hat{\imath} - 4.0\,\hat{\jmath} + 2.0\,\hat{\mathbf{k}} \\
\vec{\mathbf{d}}_3 &= 2.0\,\hat{\imath} + 3.0\,\hat{\jmath} + 1.0\,\hat{\mathbf{k}}
\end{aligned}
$$

What results from:

- $\vec{\mathbf{d}}_1 \cdot \left(\vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3\right)$
- $\vec{\mathbf{d}}_1 \cdot \left(\vec{\mathbf{d}}_2 \times \vec{\mathbf{d}}_3\right)$
- $\vec{\mathbf{d}}_1 \times \left(\vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3\right)$

Given: Three vectors $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$, and $\vec{\mathbf{d}}_3$ in two-dimensional cartesian coordinates.

Find: The result of various sums and scalar and vector products given above.

Sketch:
Relevant equations: In this case, we need only the requisite formulas for adding two vectors and taking the scalar and vector products of two vectors. Given two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$,

$$
\begin{aligned}
\vec{\mathbf{a}} &= a_x\,\hat{\imath} + a_y\,\hat{\jmath} + a_z\,\hat{\mathbf{k}} \\
\vec{\mathbf{b}} &= b_x\,\hat{\imath} + b_y\,\hat{\jmath} + b_z\,\hat{\mathbf{k}}
\end{aligned}
$$

$$\text{Then}\qquad \vec{\mathbf{a}}+\vec{\mathbf{b}} = (a_x+b_x)\,\hat{\boldsymbol{\imath}} + (a_y+b_y)\,\hat{\boldsymbol{\jmath}} + (a_z+b_z)\,\hat{\mathbf{k}}$$

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = a_xb_x + a_yb_y + a_zb_z$$

$$\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \det\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} = (a_yb_z - a_zb_y)\,\hat{\mathbf{x}} + (a_zb_x - a_xb_z)\,\hat{\mathbf{y}} + (a_xb_y - a_yb_x)\,\hat{\mathbf{z}}$$

The only other thing we need remember is to work first inside the parenthesis. For example, for the first quantity, we need to find $\vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3$ first, and then calculate the scalar product of it with $\vec{\mathbf{d}}_1$.

Symbolic solution:
We can first find the results in a purely symbolic fashion by defining

$$\vec{\mathbf{d}}_1 = -3.0\,\hat{\boldsymbol{\imath}} + 3.0\,\hat{\boldsymbol{\jmath}} + 2.0\,\hat{\mathbf{k}} = d_{1x}\,\hat{\boldsymbol{\imath}} + d_{1y}\,\hat{\boldsymbol{\jmath}} + d_{1z}\,\hat{\mathbf{k}}$$

and similarly for $\vec{\mathbf{d}}_2$ and $\vec{\mathbf{d}}_3$. Finding the answer symbolically first has the advantage of being more amenable to double-checking our work later on ...though it will require a bit more algebra in the intermediate steps. So it goes.

Starting with the first quantity, and working first inside the parenthesis:

$$\begin{aligned}
\vec{\mathbf{d}}_1\cdot\left(\vec{\mathbf{d}}_2+\vec{\mathbf{d}}_3\right) &= \vec{\mathbf{d}}_1\cdot\left[(d_{2x}+d_{3x})\,\hat{\boldsymbol{\imath}} + (d_{2y}+d_{3y})\,\hat{\boldsymbol{\jmath}} + (d_{2z}+d_{3z})\,\hat{\mathbf{k}}\right]\\
&= \left[d_{1x}\,\hat{\boldsymbol{\imath}} + d_{1y}\,\hat{\boldsymbol{\jmath}} + d_{1z}\,\hat{\mathbf{k}}\right]\cdot\left[(d_{2x}+d_{3x})\,\hat{\boldsymbol{\imath}} + (d_{2y}+d_{3y})\,\hat{\boldsymbol{\jmath}} + (d_{2z}+d_{3z})\,\hat{\mathbf{k}}\right]\\
&= d_{1x}\,(d_{2x}+d_{3x}) + d_{1y}\,(d_{2y}+d_{3y}) + d_{1z}\,(d_{2z}+d_{3z})
\end{aligned}$$

For the second quantity, we first need to calculate the cross product of the second and third vectors. It is a bit messy, but brute force is really the only way forward.

$$\begin{aligned}
\vec{\mathbf{d}}_1\cdot\left(\vec{\mathbf{d}}_2\times\vec{\mathbf{d}}_3\right) &= \vec{\mathbf{d}}_1\cdot\left[(d_{2y}d_{3z}-d_{2z}d_{3y})\,\hat{\boldsymbol{\imath}} + (d_{2z}d_{3x}-d_{2x}d_{3z})\,\hat{\boldsymbol{\jmath}} + (d_{2x}d_{3y}-d_{2y}d_{3x})\,\hat{\mathbf{k}}\right]\\
&= \left[d_{1x}\,\hat{\boldsymbol{\imath}} + d_{1y}\,\hat{\boldsymbol{\jmath}} + d_{1z}\,\hat{\mathbf{k}}\right]\cdot\left[(d_{2y}d_{3z}-d_{2z}d_{3y})\,\hat{\boldsymbol{\imath}} + (d_{2z}d_{3x}-d_{2x}d_{3z})\,\hat{\boldsymbol{\jmath}} + (d_{2x}d_{3y}-d_{2y}d_{3x})\,\hat{\mathbf{k}}\right]\\
&= d_{1x}\,(d_{2y}d_{3z}-d_{2z}d_{3y}) + d_{1y}\,(d_{2z}d_{3x}-d_{2x}d_{3z}) + d_{1z}\,(d_{2x}d_{3y}-d_{2y}d_{3x})
\end{aligned}$$

The third quantity is no more difficult; this time we first perform the addition, and then take a cross product:

$$\begin{aligned}
\vec{\mathbf{d}}_1\times\left(\vec{\mathbf{d}}_2+\vec{\mathbf{d}}_3\right) &= \vec{\mathbf{d}}_1\times\left[(d_{2x}+d_{3x})\,\hat{\boldsymbol{\imath}} + (d_{2y}+d_{3y})\,\hat{\boldsymbol{\jmath}} + (d_{2z}+d_{3z})\,\hat{\mathbf{k}}\right]\\
&= \left[d_{1y}\,(d_{2z}+d_{3z}) - d_{1z}\,(d_{2y}+d_{3y})\right]\,\hat{\boldsymbol{\imath}} + \left[d_{1z}\,(d_{2x}+d_{3x}) - d_{1x}\,(d_{2z}+d_{3z})\right]\,\hat{\boldsymbol{\jmath}}\\
&\qquad + \left[d_{1x}\,(d_{2y}+d_{3y}) - d_{1y}\,(d_{2x}+d_{3x})\right]\,\hat{\mathbf{k}}
\end{aligned}$$

There is not much point in simplifying further, there are no like terms to collect.

Numeric solution:
All that is needed now is to plug in the actual numbers, noting that $d_{1x}=-3.0$, $d_{1y}=3.0$, $d_{1z}=2.0$, *etc.* For the first quantity:

$$
\begin{aligned}
\vec{\mathbf{d}}_1 \cdot \left(\vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3\right) &= d_{1x}\left(d_{2x}+d_{3x}\right) + d_{1y}\left(d_{2y}+d_{3y}\right) + d_{1z}\left(d_{2z}+d_{3z}\right) \\
&= -3.0\left(-2.0+2.0\right) + 3.0\left(-4.0+3.0\right) + 2.0\left(2.0+1.0\right) = 0 - 3.0 + 6.0 = 3.0
\end{aligned}
$$

For the second quantity:

$$
\begin{aligned}
\vec{\mathbf{d}}_1 \cdot \left(\vec{\mathbf{d}}_2 \times \vec{\mathbf{d}}_3\right) &= d_{1x}\left(d_{2y}d_{3z} - d_{2z}d_{3y}\right) + d_{1y}\left(d_{2z}d_{3x} - d_{2x}d_{3z}\right) + d_{1z}\left(d_{2x}d_{3y} - d_{2y}d_{3x}\right) \\
&= -3.0\left(-4.0-6.0\right) + 3.0\left(4.0+2.0\right) + 2.0\left(-6.0+8.0\right) = 30 + 18.0 + 4.0 = 52.0
\end{aligned}
$$

For the third quantity:

$$
\begin{aligned}
&\left[d_{1y}\left(d_{2z}+d_{3z}\right) - d_{1z}\left(d_{2y}+d_{3y}\right)\right]\hat{\boldsymbol{\imath}} + \left[d_{1z}\left(d_{2x}+d_{3x}\right) - d_{1x}\left(d_{2z}+d_{3z}\right)\right]\hat{\boldsymbol{\jmath}} \\
&\qquad + \left[d_{1x}\left(d_{2y}+d_{3y}\right) - d_{1y}\left(d_{2x}+d_{3x}\right)\right]\hat{\mathbf{k}} \\
&= \left[3.0\left(2.0+1.0\right) - 2.0\left(-4.0+3.0\right)\right]\hat{\boldsymbol{\imath}} + \left[2.0\left(-2.0+2.0\right) + 3.0\left(2.0+1.0\right)\right]\hat{\boldsymbol{\jmath}} \\
&\qquad + \left[-3.0\left(-4.0+3.0\right) - 3.0\left(-2.0+2.0\right)\right]\hat{\mathbf{k}} \\
&= \left[9.0+2.0\right]\hat{\boldsymbol{\imath}} + \left[0+9.0\right]\hat{\boldsymbol{\jmath}} + \left[3.0+0\right]\hat{\mathbf{k}} \\
&= 11.0\,\hat{\boldsymbol{\imath}} + 9.0\,\hat{\boldsymbol{\jmath}} + 3.0\,\hat{\mathbf{k}}
\end{aligned}
$$

Double check: *Units. Order-of-magnitude.*
There are no units in this problem, but we can decide what sort of solution should we expect *qualitatively* – should the answers be vectors, scalars, or neither?

For the first quantity, the quantity inside parenthesis is the sum of two vectors, and therefore a vector itself. We then need to find the scalar product of this vector with $\vec{\mathbf{d}}_1$. The final quantity, then, is the scalar product of two vectors, which is itself a scalar (*i.e.,* just a number). This also means that the product should have only terms with the product of two components, such as $d_{1x}d_{2x}$, which is consistent with our answer.

The second quantity is similarly a scalar, since the cross product in parenthesis results in an (axial) vector, whose scalar product with $\vec{\mathbf{d}}_1$ also gives a scalar. Since there are two products here, the final answer should have only terms with three components, such as $d_{1x}d_{2y}d_{3z}$, consistent with our answer.

The third quantity has a vector resulting in the parenthesis, and we need its vector product with $\vec{\mathbf{d}}_1$, which gives an (axial) vector. Thus, only the third quantity is a vector at all, and only a pseudovector at that, the other two are just numbers. Again, we have only one product here, so the final answer should again have terms with two components.