UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 125 / LeClair Spring 2009

Problem Set 1: Solutions 3

Problems due 16 January 2009.

1. A hoodlum throws a stone vertically downward with an initial speed of 12.0 m/s from the roof of a building 30.0 m above the ground. How long does it take the stone to reach the ground, and what is its speed on impact?

Given: the initial velocity and position of a stone undergoing free-fall motion.

Find: The time required for the stone to reach the ground, 30 m below its starting point, and its speed on impact. The stone is thrown straight down, after which it is only under the influence of gravity. Thus, the motion will be along a straight (vertical) line, with constant acceleration.

Sketch: The situation is quite simple, but we can take this as an opportunity to choose a coordinate system and origin. Since the motion is one dimensional, we need only one axis. Let that be an x axis, running vertically, with the $+x$ direction being upward. We will choose the origin to be the ground level, which makes the stone's initial position $x_i = 30.0$ m. With these choices, the stone's initial velocity v_i is in the $-x$ direction, as is the acceleration due to gravity. For completion, let $t=0$ be the time at which the stone is thrown.

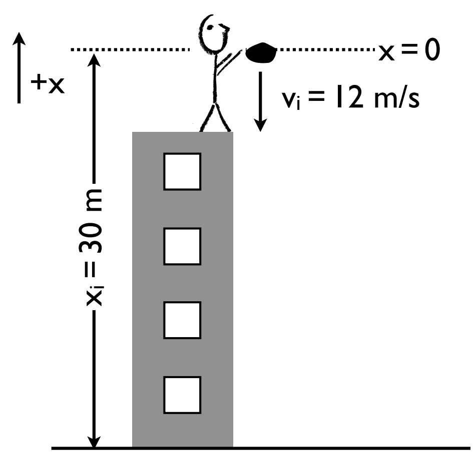

Figure 1: *Hoodlum throwing a stone off the roof of a tall building.*

Relevant equations: We derived a general equation for one-dimensional motion with constant acceleration, this is all we need along with the initial conditions. Let $x(t)$ be the stone's position at a time t, x_i its initial position, v_i its initial velocity, and a the acceleration of the stone. Then

$$x(t) = x_i + v_i t + \frac{1}{2} a t^2 \tag{1}$$

Additionally, we can get the stone's velocity (speed) by differentiating with respect to time:

$$v(t) = \frac{dx}{dt} = v_i + at \tag{2}$$

Symbolic solution: With our choice of origin, we want to find the time at which the stone reaches position $x\!=\!0$. From Eq. 1:

$$x(t) = 0 = x_i + v_i t + \frac{1}{2} a t^2 \tag{3}$$

This is just a parabolic equation in t, for which the solution is well-known:

$$t_{x=0} = \frac{-v_i \pm \sqrt{v_i^2 - 2ax_i}}{a} \tag{4}$$

We also want to find the velocity at this time, which means plugging $t_{x=0}$ into Eq. 2.

Numeric solution: With our choice of origin, and with $+x$ being in the vertical direction, we have the following boundary conditions:

$$\begin{aligned} x_i &= 30.0\,\mathrm{m} \\ v_i &= -12.0\,\mathrm{m/s} \\ a &= -g = -9.81\,\mathrm{m/s^2} \end{aligned}$$

Using these values in our solution above, the time required for the stone to reach the ground is

$$\begin{aligned} t_{x=0} &= \frac{-v_i \pm \sqrt{v_i^2 - 2ax_i}}{a} = \frac{12.0\,\mathrm{m/s} \pm \sqrt{12.0^2\,\mathrm{m^2/s^2} + 2\,(9.81\,\mathrm{m/s^2})\,(30.0\,\mathrm{m})}}{-9.81\,\mathrm{m/s^2}} \\ &\approx \{1.54\,\mathrm{s}, -3.99\,\mathrm{s}\} \end{aligned}$$

We reject the negative root as unphysical – this solution is a time before the ball was thrown, which makes no sense. This solution is *mathematically* valid, but our problem is physically meaningful only for $t > 0$. Taking the positive root as our solution, the ball hits the ground about $1.54\,\mathrm{s}$ after it is thrown.

At the point the ball hits the ground, the velocity is

$$v(1.54\,\mathrm{s}) = v_i + at \approx -12.0\,\mathrm{m/s} - \left(9.8\,\mathrm{m/s^2}\right)(1.54\,\mathrm{s}) = -27.1\,\mathrm{m/s}$$

Double check: If we ignore the acceleration due to gravity, at an initial velocity of $12.0\,\mathrm{m/s}$, the stone should cover $30.0\,\mathrm{m}$ in $2.5\,\mathrm{s}$ ($x\!=\!v_i t$). On the other hand, if we ignore the initial velocity, and consider only free-fall motion, it should take about $1.75\,\mathrm{s}$ ($x = \frac{1}{2} g t^2$). The real answer should be slightly less time than either of these, since we are combining the effects of gravity and the initial velocity, both of which act in the same direction. Our answer of $1.5\,\mathrm{s}$ seems reasonable in this light.

We can also check the units of Eq. 4 with dimensional analysis to be sure:

$$t_{x=0} = \frac{[\mathrm{m/s}] \pm \sqrt{[\mathrm{m/s}]^2 - 2[\mathrm{m/s^2}][\mathrm{m}]}}{\mathrm{m/s^2}} = \frac{[\mathrm{m/s}] \pm \sqrt{[\mathrm{m^2/s^2}]}}{\mathrm{m/s^2}} = \frac{[\mathrm{m/s}] \pm [\mathrm{m/s}]}{\mathrm{m/s^2}} = [\mathrm{s}]$$

The units are seconds, as we require.

2. The position of a particle moving along the x axis is given in centimeters by

$$x = 9.75 + 1.50t^3$$

where t is in seconds. Calculate the instantaneous velocity and acceleration at $t = 2.50$ s.

Given: position versus time $x(t)$ in meters and seconds.

Find: Instantaneous velocity and acceleration at $t = 2.50$ s.

Sketch: It is somewhat helpful to graph the position versus time. Most graphing programs these days will also calculate derivatives for you, so it is little extra work to plot velocity and acceleration as well.

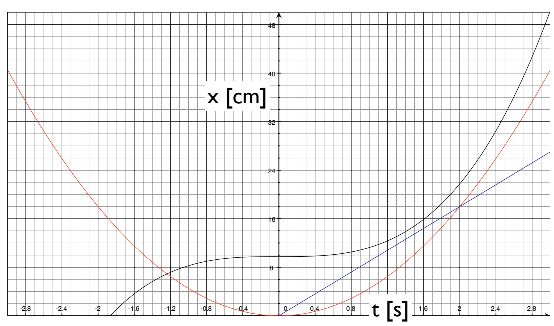

Figure 2: *Position (black), velocity (red), and acceleration (blue) as a function of time.*

Relevant equations: We need only the definitions of instantaneous velocity and acceleration:

$$v(t) = \frac{d}{dt}x(t) \tag{5}$$

$$a(t) = \frac{d}{dt}v(t) = \frac{d^2}{dt^2}x(t) \tag{6}$$

Symbolic solution:

$$v(t) = \frac{d}{dt}\left(9.75 + 1.50t^3\right) = 4.50t^2 \tag{7}$$

$$a(t) = \frac{d}{dt}\left(4.50t^2\right) = 9t \tag{8}$$

Numeric solution: Evaluating the equations above at $t = 2.50\,\mathrm{s}$,

$$v(2.50\,\mathrm{s}) \approx 28.1\,\mathrm{cm/s}$$
$$a(2.50\,\mathrm{s}) \approx 22.5\,\mathrm{cm/s^2}$$

Double check: We can simply read the values off of the graph above, which gives consistent results.

Dimensional analysis on the $x(t)$ equation given will tell us what the units of the constants must be, which will let us check our answer. All terms must have units of meters. Thus, the constant 9.75 must have units of cm, while the constant 1.5 must have units cm/s^3, since it is multiplied by t^3 in s^3 and results in m. Equation 7 has then a constant in cm/s^3 multiplied by t^2 in s^2, giving cm/s, while Eq. 8 has a constant in cm/s^3 multiplied by t in s, giving cm/s^2. Both have the correct units.

3. Two trains are 100 km apart on the same track, headed on a collision course towards each other. Both are traveling 50 km per hour. A very speedy bird takes off from the first train and flies at 75 km per hour toward the second train. The bird then immediately turns around and flies back to the first train. Then he flies back to the second train, and repeats the process over and over as the distance between the trains diminishes. How far will he have flown before the trains collide?

This is a rather famous problem, which has a very simple solution. The two trains are going toward each other at 50 km/h, meaning they close the distance between them at a rate of 100 km/h. Thus, they will meet in precisely 1 hr. The bird flies then for a total of one hour at 75 km/h, and hence it must cover a distance of 75 km.

There is also a more painful infinite series solution to this problem. See, for example,

`http://scienceblogs.com/builtonfacts/2008/12/bouncing_birds.php`

for some background and discussion on this problem.