UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 125 / LeClair   Spring 2009

Problem Set 2 Solutions

The following three problems are due 20 January 2009 at the beginning of class.

1. (H,R,&W 4.39) A rifle that shoots a bullet at $460\,\mathrm{m/s}$ is to be aimed at a target $45.7\,\mathrm{m}$ away. If the center of the target is level with the rifle, how high above the target must the rifle barrel be pointed so that the bullet hits dead center?

Given: The magnitude of the initial velocity of a fired bullet v_i and its distance from a target d.

Find: The height above the target that the shooter must aim y_{aim}. We can easily find this once we know the firing angle θ required for the bullet to hit the target. That is, the angle such that the bullet is at the same height a distance d from where it is fired.

Sketch: For convenience, let the origin be at the position the bullet is fired from. Let the $+x$ axis run horizontally, from the bullet to the target, and let the $+y$ axis run vertically. Let time $t=0$ be the moment the projectile is launched.

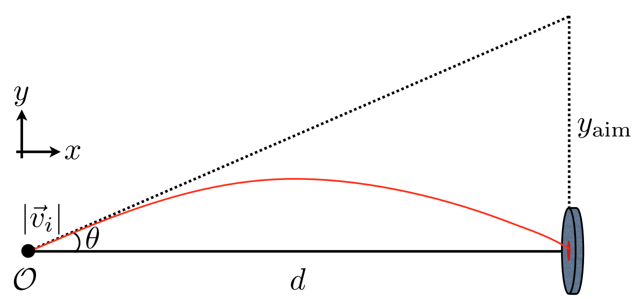


Figure 1: *Firing a rifle at a distant target. The bullet's trajectory is (approximately) shown in red.*

The bullet is fired at an initial velocity $|\vec{\mathbf{v}}_i|$ and angle θ, a distance d from a target. The target is at the same vertical position as the rifle, so we need to find the angle θ and resulting y_{aim} such that the bullet is at $y=0$ at $x=d$.

Relevant equations: In the x direction, we have constant velocity and no acceleration, with position starting at the origin at $t=0$:

$$x(t) = v_{ix}t = |\vec{\mathbf{v}}_i|t\cos\theta \tag{1}$$

In the y direction, we have an initial constant velocity of $v_{iy} = |\vec{\mathbf{v}}_i|\sin\theta$ and a constant acceleration of $a_y = -g$:

$$y(t) = v_{iy}t - \frac{1}{2}gt^2 \tag{2}$$

Solving Eq. 1 for t and substituting into Eq. 2 yields our general projectile equation, giving the path of the projectile $y(x)$ when launched from the origin with initial velocity $|\vec{\mathbf{v}}_i|$ and angle θ above the x axis:

$$y(x) = x \tan\theta - \frac{gx^2}{2|\vec{\mathbf{v}}_i|^2 \cos^2\theta} \tag{3}$$

With our chosen coordinate system and origin, $y_o = 0$. We also need the aiming height above the target in terms of the target distance and firing angle, which we can get from basic trigonometry:

$$\tan\theta = \frac{y_{\text{aim}}}{d} \tag{4}$$

Note that one can also use the "range equation" directly, but this is less instructive. It is fine for you to do this in your own solutions, but keep in mind you will probably not be given these sort of specialized equations on an exam – you should know how to derive them.

Symbolic solution: We desire the bullet to reach point $(d, 0)$. Substituting these coordinates into Eq. 3, and solving for θ:

$$\begin{aligned}
y(x) &= x \tan\theta - \frac{gx^2}{2|v_i|^2 \cos^2\theta} && (5)\\
0 &= d \tan\theta - \frac{gd^2}{2|v_i|^2 \cos^2\theta} && (6)\\
d \tan\theta &= \frac{gd^2}{2|v_i|^2 \cos^2\theta} && (7)\\
\tan\theta \cos^2\theta &= \frac{gd}{2|\vec{\mathbf{v}}_i|^2} && (8)\\
\sin\theta\cos\theta &= \frac{1}{2}\sin 2\theta = \frac{gd}{2|\vec{\mathbf{v}}_i|^2} && (9)\\
\Longrightarrow \theta &= \frac{1}{2}\sin^{-1}\left[\frac{gd}{|\vec{\mathbf{v}}_i|^2}\right] && (10)
\end{aligned}$$

Given θ, rearranging Eq. 4 gives us the aiming height:

$$y_{\text{aim}} = d \tan\theta \tag{11}$$

Numeric solution: We are given $|\vec{\mathbf{v}}_i| = 460\,\text{m/s}$ and $d = 45.7\,\text{m}$:

$$\theta = \frac{1}{2}\sin^{-1}\left[\frac{gd}{|\vec{\mathbf{v}}_i|^2}\right] = \frac{1}{2}\sin^{-1}\left[\frac{\left(9.81\,\text{m/s}^2\right)(4.57\,\text{m})}{(460\text{m/s})^2}\right] \approx 0.06069^\circ \tag{12}$$

Given the angle, we can find the height above the target we need to aim:

$$y_{\text{aim}} = d \tan\theta \approx (45.7\,\text{m}) \tan(0.06069^\circ) \xrightarrow[\text{digits}]{\text{sign.}} 0.0484\,\text{m} = 4.84\,\text{cm} \tag{13}$$

Double check: One check is use the pre-packaged projectile range equation and make sure that we get the same answer. Given $\theta \approx 0.0607^\circ$, we should calculate a range of d.

$$R = \frac{|\vec{\mathbf{v}}_i|^2 \sin 2\theta}{g} = \frac{(460\,\mathrm{m/s})\,(\sin 0.1214^\circ)}{(9.81\,\mathrm{m/s^2})} \approx 45.7\,\mathrm{m} \tag{14}$$

This is not truly an independent check, since it is derived using the same equations we used above, but it is a nice indication that we haven't gone wrong anywhere.

As a more independent estimate, we can first calculate the time it would take the bullet to reach the target in the absence of gravitational acceleration – if it were just heading straight toward the target at 460 m/s. This is not so far off the real time, since the firing angle is small anyway:

$$t_{\text{est}} = \frac{d}{|\vec{\mathbf{v}}_i|} \approx 0.1\,\mathrm{s} \tag{15}$$

In that time, how far would the bullet fall under the influence of gravity (alone)?

$$y_{\text{fall}} \approx -\frac{1}{2} g t_{\text{est}}^2 \approx 0.05\,\mathrm{m} \tag{16}$$

Thus, we estimate that the bullet should fall about 5 cm on its way to the target, meaning we should aim about 5 cm high, in line with what we calculate by more exact means.

You can also verify that units come out correctly in Eq. 12 and Eq. 13. The argument of the $\sin^{-1}$ function must be dimensionless, as it is, and y_{aim} should come out in meters, as it does. If you carry the units through the entire calculation, or at least solve the problem symbolically, without numbers until the last step, this sort of check is trivial.

2. (H,R,&W 4.17) A particle leaves the origin with an initial velocity of $\vec{\mathbf{v}} = (3.00\,\hat{\imath})$ m/s, and a constant acceleration of $\vec{\mathbf{a}} = (-1.00\,\hat{\imath} - 0.500\,\hat{\jmath})$ m/s^2. When it reaches its maximum x coordinate, what are its velocity and position vectors?

Given: The initial velocity and acceleration vectors of a particle.

Find: The maximum x coordinate, and its velocity and position vectors at the corresponding time.

Sketch: A sketch may be a bit pedantic in this case, but here you are:

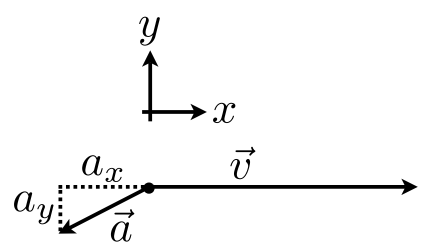


Figure 2: *A particle at the origin with an initial velocity and acceleration.*

Relevant equations: We can find the maximum x coordinate by finding the x components of the velocity and acceleration and plugging them into our derived expression for $x(t)$. This is valid because we have constant acceleration.

$$a_x = \vec{\mathbf{a}} \cdot \hat{\imath} \tag{17}$$
$$v_{ix} = \vec{\mathbf{v}} \cdot \hat{\imath} \tag{18}$$
$$x(t) = x_i + v_{ix}t + \frac{1}{2}a_x t^2 \tag{19}$$

Since our particle starts out at the origin, $x_i = 0$. Once we have $x(t)$, we can find the time at which the x coordinate is maximum, t_{max}: by differentiation:

$$\left.\frac{dx}{dt}\right|_{t_{\text{max}}} = 0 \qquad \text{and} \qquad \left.\frac{d^2x}{dt^2}\right|_{t_{\text{max}}} < 0 \tag{20}$$

Next, we need to find the position vector $\vec{\mathbf{r}}(t)$:

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t + \frac{1}{2}\vec{\mathbf{a}}\,t^2 \tag{21}$$

Again, since the particle starts at the origin, $\vec{\mathbf{r}} = 0$. Finally, we need to evaluate $\vec{\mathbf{v}}$ and $\vec{\mathbf{r}}$ at t_{max} to complete the problem.

Symbolic solution: First, we can immediately write down the equation for $x(t)$ and maximize it:

$$
\begin{aligned}
&x(t) = x_i + v_{ix}t + \frac{1}{2}a_x t^2 \\
&\left.\frac{dx}{dt}\right|_{t_{\max}} = v_{ix} + a_x t_{\max} = 0 \quad \implies \quad t_{\max} = -\frac{v_{ix}}{a_x} \\
&\frac{d^2x}{dt^2} = a_x
\end{aligned} \tag{22}
$$

The extreme time will correspond to a maximum position provided the second derivative is negative once we plug in the x components of the velocity and acceleration. Given the time of maximum x coordinate, we can write down the position and velocity vectors at that time readily:

$$
\begin{aligned}
&\vec{\mathbf{v}}(t_{\max}) = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t_{\max} = \vec{\mathbf{v}}_i - \frac{v_{ix}\vec{\mathbf{a}}}{a_x} \\
&\vec{\mathbf{r}}(t_{\max}) = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t_{\max} + \frac{1}{2}\vec{\mathbf{a}}\,t_{\max}^2 = \vec{\mathbf{r}}_i - \frac{v_{ix}\vec{\mathbf{v}}_{\mathbf{i}}}{a_x} + \frac{v_{ix}^2\vec{\mathbf{a}}}{2a_x^2}
\end{aligned} \tag{23}
$$

Numeric solution: The x components of the velocity and acceleration are

$$
\begin{aligned}
&a_x = \vec{\mathbf{a}} \cdot \hat{\imath} = -1.00\,\mathrm{m/s^2} \\
&v_x = \vec{\mathbf{v}} \cdot \hat{\imath} = 3.00\,\mathrm{m/s}
\end{aligned} \tag{24}
$$

which makes the time for the maximum x coordinate

$$
t_{\max} = -\frac{v_{ix}}{a_x} = 3\,\mathrm{s} \tag{25}
$$

Note also that $d^2x/dt^2 = -1.00$, which ensures that we have found a maximum of $x(t)$. At this point, we can just write down the expressions for $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{v}}(t)$ and plug in the numbers:

$$
\begin{aligned}
\vec{\mathbf{v}}(t_{\max}) &= \vec{\mathbf{v}}_i + \vec{\mathbf{a}}\,t_{\max} \\
&= (3.00\,\mathrm{m/s}\,\hat{\imath}) + \left(-1.00\,\mathrm{m/s^2}\,\hat{\imath} - 0.500\,\mathrm{m/s^2}\,\hat{\jmath}\right)(3.00\,\mathrm{s}) = (-1.5\,\hat{\jmath})\ \mathrm{m/s}
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\vec{\mathbf{r}}(t_{\max}) &= \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t_{\max} + \frac{1}{2}\vec{\mathbf{a}}\,t_{\max}^2 \\
&= 0 + (3.00\,\mathrm{m/s}\,\hat{\imath})\,(3.00\,\mathrm{s}) + \frac{1}{2}\left(-1.00\,\mathrm{m/s^2}\,\hat{\imath} - 0.500\,\mathrm{m/s^2}\,\hat{\jmath}\right)(3.00\,\mathrm{s})^2 \\
&= (9.00\,\mathrm{m}\,\hat{\imath}) + (-4.50\,\mathrm{m}\,\hat{\imath} - 2.25\,\mathrm{m}\,\hat{\jmath}) = (4.5\,\hat{\imath} - 2.25\,\hat{\jmath})\ \mathrm{m}
\end{aligned} \tag{27}
$$

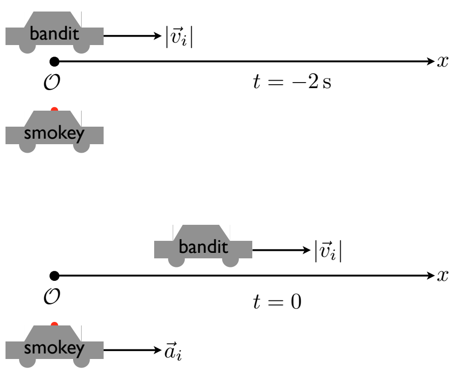
Double check: We carried the units throughout our calculations, and can be fairly confident that they are correct. We also performed the second derivative test to ensure that we found a maximum in $x(t)$.

3. A car is traveling at a constant velocity of $18\,\mathrm{m/s}$ and passes a police cruiser. Exactly $2.0\,\mathrm{s}$ after passing, the cruiser begins pursuit, with a constant acceleration of $2.5\,\mathrm{m/s^2}$. How long does it take for the cruiser to overtake the car (from the moment the cop car starts)?

Given: The constant velocity and initial position of a speeding car, the constant acceleration and initial position of a pursuing police cruiser $2.0\,\mathrm{s}$ later.

Find: How long it takes the police cruiser to overtake the car. This means we need the position versus time for each vehicle, from which we can find their intersection point.

Sketch: Let the car's and cruiser's initial position be $x=0$, with the direction of travel being the $+x$ axis. Let time $t=0$ be when *the cruiser begins accelerating*. Thus, the car starts off with $\vec{\mathbf{v}}_i=18\,\mathrm{m/s}\,\hat{\boldsymbol{\imath}}$ at $t=-2\,\mathrm{s}$ from $x=0$, and the police cruiser starts off with $\vec{\mathbf{a}}_i=2.5\,\mathrm{m/s^2}\,\hat{\boldsymbol{\imath}}$ at $t=0$, also from $x=0$.



A car chase. **Upper:** *at time $t=-2\,s$ the car passes the stationary police cruiser.* **Lower** *at time $t=0\,s$, the police cruiser begins chase with constant acceleration.*

Relevant equations: We need the general equation for position in one dimension under the conditions of constant (or zero) acceleration:

$$x(t)=x_i+v_{ix}t+\frac{1}{2}a_xt^2 \tag{28}$$

Symbolic solution: For the car, we have $x_i=0$ and $a_x=0$. Let the car's initial velocity be v_{ix}. The car has been moving at constant velocity for two seconds before the cruiser starts, so we need to shift the time coordinate from $t\mapsto(t+2)$.

$$x_{\text{car}}=v_{ix}\,(t+2\,\mathrm{s}) \tag{29}$$

You can verify that this correctly gives $x_{\text{car}}\,(-2\,\mathrm{s})=0$. In order to keep things more general, however, let us say that the cruiser starts δt seconds later (we can later set $\delta t=2\,\mathrm{s}$):

$$x_{\text{car}}=v_{ix}\,(t+\delta t) \tag{30}$$

For the police cruiser, we have $x_i=0$ and $v_{ix}=0$. Let the cruiser's initial acceleration be a_x. The cruiser starts out at $t=0$, so things are simple:

$$x_{\text{cop}}=\frac{1}{2}a_xt^2 \tag{31}$$

We need to find the time t at which $x_{\text{car}} = x_{\text{cop}}$, from which we can get the time it takes for the police cruiser to overtake the car, *viz.* $t - 2$.

$$
\begin{align}
x_{\text{car}} &= x_{\text{cop}} \tag{32}\\
v_{ix}\left(t+\delta t\right) &= \frac{1}{2}a_x t^2 \tag{33}\\
v_{ix}t + \delta t\, v_{ix} &= \frac{1}{2}a_x t^2 \tag{34}\\
0 &= a_x t^2 - 2v_{ix}t - 2\delta t\, v_{ix} \tag{35}\\
\implies \quad t &= \frac{2v_{ix} \pm \sqrt{4v_{ix}^2 + 8\delta t\, v_{ix}a_x}}{2a_x} \tag{36}
\end{align}
$$

It is the "+" solution in the $\pm$ we want, as our numerical solution below will verify.

Numeric solution: Using the numbers we are given:

$$
\begin{align}
t = \frac{2v_{ix} \pm \sqrt{4v_{ix}^2 + 8\delta t\, v_{ix}a_x}}{2a_x} &= \frac{2\left(18\,\text{m/s}\right) \pm \sqrt{4\left(18\,\text{m/s}\right)^2 + 8\left(2\,\text{s}\right)\left(18\,\text{m/s}\right)\left(2.5\,\text{m/s}^2\right)}}{2\left(2.5\,\text{m/s}^2\right)} \notag\\
= \frac{36\,\text{m/s} \pm \sqrt{1296\,\text{m}^2/\text{s}^2 + 720\,\text{m}^2/\text{s}^2}}{5\,\text{m/s}^2} &= \frac{\left(36 \pm 44.9\right)\ \text{m/s}}{5\,\text{m/s}^2} = \left\{16.2, -1.78\right\}\ \text{s} \tag{37}
\end{align}
$$

The negative solution we can reject as unphysical, so it takes approximately 16 s for the cruiser to overtake the car.

Double check: We carried units throughout the numerical phase of the calculation, and our end result comes out in seconds as required. Another check we can make is to calculate the position of both the car and cruiser at the time we found above to be sure they are the same:

$$
\begin{align}
x_{\text{car}} &= v_{ix}\left(t+\delta t\right) = \left(18\,\text{m/s}\right)\left(18.2\,\text{s}\right) \approx 327.6\,\text{m} \xrightarrow[\text{digits}]{\text{sign.}} 330\,\text{m} \tag{38}\\
x_{\text{cop}} &= \frac{1}{2}a_x t^2 = \frac{1}{2}\left(2.5\,\text{m/s}^2\right)\left(16.2\,\text{s}\right)^2 \approx 328.05\,\text{m} \xrightarrow[\text{digits}]{\text{sign.}} 330\,\text{m} \tag{39}
\end{align}
$$

Within the precision implied by the number of significant digits, the two positions are the same; our answer is reasonable.

The following three problems are due 22 January 2009 at the beginning of class.

4. A projectile is launched with initial velocity $\vec{\mathbf{v}}_i$ and angle θ a distance d from a ramp inclined at angle φ (see figure below). What is the constraint on the initial velocity and angle for the projectile to hit the ramp (*i.e.*, it does not fall short)? *No numerical solution is required.*

Given: The initial velocity $\vec{\mathbf{v}}_i$ and its angle θ with respect to the x axis for a projectile and the distance to a ramp d.

Find: A constraint on $\vec{\mathbf{v}}_i$ and θ for the projectile to hit the ramp. Put another way, we want to find

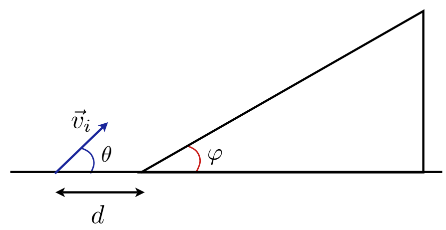


Figure 3: *Problems 4 and 5: a projectile is launched with initial velocity $\vec{\mathbf{v}}_i$ and angle θ a distance d from a ramp inclined at angle φ.*

the conditions under which the range of the projectile is greater than or equal to the distance from the ramp d.

Sketch: The figure above will do nicely.

Relevant equations: Since we essentially derived it above, we can use the equation for the range of a projectile:

$$R = \frac{|\vec{\mathbf{v}}_i|^2 \sin 2\theta}{g} \tag{40}$$

We want the range R to equal or exceed d, $R \geq d$.
Symbolic solution: There is not much to it. Set $R \geq d$ and solve for the product of $\sin 2\theta$ and $|\vec{\mathbf{v}}_i|^2$.

$$\begin{aligned} R &= \frac{|\vec{\mathbf{v}}_i|^2 \sin 2\theta}{g} \geq d \\ |\vec{\mathbf{v}}_i|^2 \sin 2\theta &\geq gd \end{aligned} \tag{41}$$

Numeric solution: Not required. There are no numbers given anyway.

Double check: Both sides of the equation above should have the same units. The units of the right side are simply those of velocity squared, $\mathrm{m}^2/\mathrm{s}^2$, since sin is dimensionless. The units of the right side are $(\mathrm{m}/\mathrm{s}^2)(\mathrm{m}) = \mathrm{m}^2/\mathrm{s}^2$, so our equation is dimensionally correct.

We can also qualitatively check the dependencies of Eq. 41. For instance, if we fix θ, we expect that a larger initial velocity will make the condition easier to fulfill (more likely to hit the ramp). This is borne out by Eq. 41 – if $|\vec{\mathbf{v}}_i|$ increases, the inequality is more likely to be true. Similarly, if we increase d, we expect the condition to be less likely, consistent with Eq. 41. Finally, if we were to increase the gravitational acceleration g (clearly we cannot in reality), it should be harder to hit the ramp, also consistent with Eq. 41

5. Referring to the preceding problem, how far along the ramp (laterally), and at what height, does the projectile hit the ramp? You may assume that the ramp is incredibly long. *No numerical solution is required.*

Find: The point at which a launched projectile hits a ramp inclined at angle φ a distance d from the ramp.

Given: The projectile's launch speed and angle, the distance to the ramp, and the ramp angle.

Sketch: Let the origin be at the projectile's launch position, with the x and y axes of a cartesian coordinate system aligned as shown below.

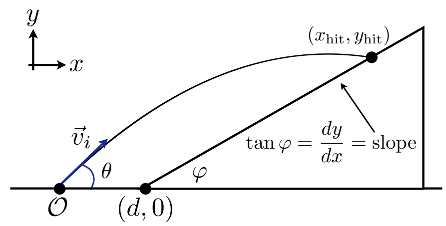


Figure 4: *Problems 5: Where does the projectile hit the ramp?*

Thus ramp begins at position $(d, 0)$, and the projectile is launched from $(0, 0)$. We seek the intersection of the projectile's trajectory with the surface of the ramp at position $(x_{\text{hit}}, y_{\text{hit}})$, subject to the condition that $y_{\text{hit}} \geq 0$, *i.e.,* the projectile actually reaches the ramp.

Relevant equations: We have already derived the trajectory $y(x)$ for a projectile launched from the origin:

$$y_p = x \tan\theta - \frac{gx^2}{2|\vec{\mathbf{v}}_i|^2 \cos^2\theta} \tag{42}$$

The ramp itself can be described by a simple line. We know the slope m of the ramp is $m = \Delta x / \Delta y = \tan\varphi$, and we know it intersects the point $(x_o, y_o) = (d, 0)$. This is sufficient to derive an equation of the line describing the ramp's surface, $y_r(x)$, using point-slope form:

$$\begin{aligned} y_r - y_o &= m\,(x - x_o) \\ y_r &= (\tan\varphi)\,(x - d) \end{aligned}$$

We also need a condition to check that the projectile actually hits the ramp, which means we require $y_{\text{hit}} > 0$ and correspondingly, $x_{\text{hit}} > d$. Finally, the point of intersection must occur when $y_r = y_p \equiv y_{\text{hit}}$.

Symbolic solution: We need only impose the condition $y_r = y_p$ to begin our solution. The resulting x value is the x_{hit} we desire

$$
\begin{aligned}
y_r = (\tan\varphi)\,(x-d) = y_p &= x\tan\theta - \frac{gx^2}{2|\vec{\mathbf{v}}_i|^2\cos^2\theta} \\
0 &= \left[\frac{g}{2|\vec{\mathbf{v}}_i|^2\cos^2\theta}\right]x^2 + \left[\tan\varphi - \tan\theta\right]x - d\tan\varphi \\
\Longrightarrow \qquad x_{\text{hit}} &= \frac{\left[\tan\theta - \tan\varphi\right] \pm \sqrt{\left[\tan\varphi - \tan\theta\right]^2 + \dfrac{2d\tan\varphi}{|\vec{\mathbf{v}}_i|^2\cos^2\theta}}}{\dfrac{g}{|\vec{\mathbf{v}}_i|^2\cos^2\theta}}
\end{aligned} \tag{43}
$$

There is really not much simplification we can do here.

$$
x_{\text{hit}} = \left[\frac{|\vec{\mathbf{v}}_i|^2\cos^2\theta}{g}\right]\left[\left[\tan\theta - \tan\varphi\right] \pm \sqrt{\left[\tan\varphi - \tan\theta\right]^2 + \frac{2d\tan\varphi}{|\vec{\mathbf{v}}_i|^2\cos^2\theta}}\right] \tag{44}
$$

One can use the identity

$$
\tan\theta - \tan\varphi = \frac{\sin\left(\theta - \varphi\right)}{\cos\theta\cos\varphi}
$$

but the result is no prettier than what we have. We have two problems remaining now: which of the two roots do we take, and what is the corresponding y coordinate y_{hit}? The former problem is most straightforwardly solved by taking the limit $\varphi \to 0$, that is, getting rid of the ramp entirely. In that case, what we have really solved for is the x coordinate where the projectile would hit the ground – the range of the projectile over level ground.

$$
\begin{aligned}
x_{\text{hit}}\big|_{\varphi\to 0} &= \left[\frac{|\vec{\mathbf{v}}_i|^2\cos^2\theta}{g}\right]\left[\tan\theta \pm \sqrt{\tan^2\theta}\right] = \left[\frac{|\vec{\mathbf{v}}_i|^2\cos^2\theta}{g}\right]\left[\tan\theta \pm \tan\theta\right] \\
&= \left\{0, \frac{2|\vec{\mathbf{v}}_i|^2\cos^2\theta}{g}\tan\theta\right\} = \left\{0, \frac{2|\vec{\mathbf{v}}_i|^2\sin\theta\cos\theta}{g}\right\} = \left\{0, \frac{|\vec{\mathbf{v}}_i|^2\sin 2\theta}{g}\right\}
\end{aligned} \tag{45}
$$

The positive root of Eq. 44 recovers the previously derived range formula in the limit $\varphi \to 0$, so this is the solution we seek (the negative root gives us the other position where $y=0$, the starting position). Thus,

$$
x_{\text{hit}} = \left[\frac{|\vec{\mathbf{v}}_i|^2\cos^2\theta}{g}\right]\left[\left[\tan\theta - \tan\varphi\right] + \sqrt{\left[\tan\varphi - \tan\theta\right]^2 + \frac{2d\tan\varphi}{|\vec{\mathbf{v}}_i|^2\cos^2\theta}}\right] \tag{46}
$$

The y coordinate where the projectile hits the ramp is then most easily found from the ramp equation:

$$
\begin{aligned}
y_{\text{hit}} &= y_r(x_{\text{hit}}) = (\tan\varphi)\,(x_{\text{hit}} - d) \\
&= \tan\varphi\left[\left[\frac{|\vec{\mathbf{v}}_i|^2\cos^2\theta}{g}\right]\left[[\tan\theta - \tan\varphi] + \sqrt{[\tan\varphi - \tan\theta]^2 + \frac{2d\tan\varphi}{|\vec{\mathbf{v}}_i|^2\cos^2\theta}}\right] - d\right] \\
&= \left[\frac{|\vec{\mathbf{v}}_i|^2\cos^2\theta\tan\varphi}{g}\right]\left[[\tan\theta - \tan\varphi] + \sqrt{[\tan\varphi - \tan\theta]^2 + \frac{2d\tan\varphi}{|\vec{\mathbf{v}}_i|^2\cos^2\theta}}\right] - d\tan\varphi
\end{aligned}
\tag{47}
$$

This is even more of a monster. Once again, check the limit $\varphi \to 0$: in that case, we simply get $y_{\text{hit}} = 0$, as we would expect. Since $\varphi = 0$ corresponds to there being no ramp at all, what we have really found in that limit is the y coordinate where the projectile hits the level ground, which we chose as $y = 0$.

Finally, this solution is subject to our condition $x_{\text{hit}} > d$. Any value for x_{hit} smaller than d will result in the projectile not reaching the ramp at all (and will give a negative value for y_{hit}. Equivalently, we can also apply our constrain from problem 4 above.

Numeric solution: N/A.

Double check: We have already checked the limit $\varphi \to 0$ above, and found a sensible result, *viz.*, we recover the "range" equation. Another interesting limit is $d \to 0$, where we launch right at the start of the ramp. In that case,

$$
x_{\text{hit}}|_{d\to 0} = \frac{2|\vec{\mathbf{v}}_i|^2\cos^2\theta}{g}\left[\tan\theta - \tan\varphi\right] \tag{48}
$$

In this case, the projectile only goes forward ($x_{\text{hit}} > 0$) if the launch angle exceeds the ramp angle ...which makes some sense. If $\theta < \varphi$, we would be launching the projectile *inside* the ramp! You can easily verify now that under both limits $\varphi \to 0$ and $d \to 0$ the range equation is again recovered.

Since there are no numbers in this problem, we cannot make an order of magnitude estimate. The best we can do is investigate various limits, and check that the behavior of our equations makes sense.

6. (H,R,&W 4.53) A ball rolls horizontally off the top of a stairway with a speed of $1.52\,\text{m/s}$. The steps are $20.3\,\text{cm}$ high and $20.3\,\text{cm}$ wide. Which step does the ball hit first. You may assume that there are many, many stairs.

Given: The dimensions of a staircase, and the initial velocity of a ball which rolls off the staircase. Let the staircase width and height be d, and the ball's initial speed $|\vec{\mathbf{v}}_i|$.

Find: Which step the ball first hits on its way down.

Sketch: We choose a cartesian coordinate system with x and y axes aligned with the stairs, as shown below. Let the origin be the point at which the ball leaves the topmost stair. The ball is launched horizontally off of the top step, and will follow a parabolic trajectory down the staircase.
How do we determine which stair will first be hit? From the sketch, it is clear that we need to find at which point the ball's parabolic trajectory (solid curve) passes below a line connecting the right-most

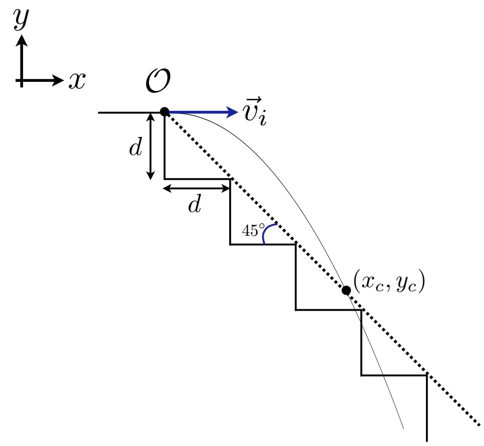


Figure 5: *Problem 6: A projectile launched off the top of a staircase.*

tips of each stair (dotted line).

Relevant equations: Based on our logic above, we need an equation for the ball's trajectory and an equation for the line describing the staircase boundary. The staircase itself is is composed of steps of equal height and width. Therefore, a line from the origin connecting the right-most tip of each stair (the dotted line in the figure) will have a slope of -1, and can be described by $y_s = -x$.

The ball will follow our now well-known parabolic trajectory. In this case, the launch angle is zero, and the ball's motion is described by setting $\theta = 0$ in Eq. 3:

$$y_b = -\frac{gx^2}{2|\vec{\mathbf{v}}_i|^2} \tag{49}$$

We first need to find the x coordinate where $y_b = y_x$, which is the point where the parabolic trajectory dips below the line defining the staircase slope. Call this coordinate x_c. Given this coordinate, we need to determine how many stairs this distance corresponds to. The ratio of x_c to the stair width should give us this number. However, we must keep in mind the fact that the staircase is discrete: if we find that x_c corresponds to, for example, 4.7 stair widths, what does that mean? It means the ball crossed the fourth stair, but 70% of the way across the fifth one, its trajectory dipped below the line defining the staircase. Thus, the ball would hit the fifth stair.

What we need, then, is to *find the ratio of x_c and the stair width d, and take the next largest integer*. This gives us the number of the stair the ball first hits n_s. There is a mathematical function that does exactly what we want, for this operation, the *ceiling* function. It takes a real-valued argument x and gives back the next-highest integer. For example, if $x = 3.2$, then the ceiling of x is 4. The standard notation is $\lceil x \rceil = 4$.[i].

$$n_s = \left\lceil \frac{x_c}{d} \right\rceil$$

[i]If you want to get all technical, $\lceil x \rceil = \min\{n \in \mathbb{Z} \,|\, n \geq x\}$

Symbolic solution: First, we need to find the point x_c where the ball's parabolic trajectory intersects the staircase boundary line:

$$
\begin{aligned}
y_b &= -\frac{gx^2}{2|v_i|^2} = y_s = -x \\
0 &= \frac{gx^2}{2|\vec{\mathbf{v}}_i|^2} - x \\
0 &= x\left[\frac{gx}{2|\vec{\mathbf{v}}_i|^2} + 1\right] \\
\implies \quad x_c &= \left\{0, \frac{2|\vec{\mathbf{v}}_i|^2}{gd}\right\}
\end{aligned}
$$

As usual, one of our answers is the trivial solution, the one where the ball never leaves the staircase ($x_c=0$). The second solution is what we are after. The number of the stair that is first hit is then

$$n_s = \left\lceil \frac{2|\vec{\mathbf{v}}_i|^2}{gd} \right\rceil$$

Numeric solution: We are given $|\vec{\mathbf{v}}_i|^2 = 1.52\,\mathrm{m/s}$ and $d = 20.3\,\mathrm{cm} = 0.203\,\mathrm{m}$. Additionally, we need $g \approx 9.81\,\mathrm{m/s^2}$.

$$n_s = \left\lceil \frac{2|\vec{\mathbf{v}}_i|^2}{gd} \right\rceil = \left\lceil \frac{2\,(1.52\,\mathrm{m/s})^2}{(9.81\,\mathrm{m/s^2})\,(0.203\,\mathrm{m})} \right\rceil = \left\lceil 2.32 \right\rceil = 3$$

The ball will hit the third stair.

Double check: The ratio x_c/d must be dimensionless, as we have shown it to be above; our units are correct. Another "brute force" method of checking our result is to calculate the y position of the projectile at the right-most edge of successive stairs. At the right-most edge of the n^{th} stair, the x coordinate is nd. If the y coordinate of the projectile's trajectory is *below* $(-nd)$ for the right-most edge of a given stair, then we must have hit that stair.

Stair n	$x = nd$ **(m)**	$y_s = nd$ **(m)**	y_b **(m)**	**result**
1	0.203	-0.203	-0.0875	cleared
2	0.406	-0.406	-0.350	cleared
3	0.609	-0.609	-0.787	hit

The brute-force method confirms our result: the third stair is not cleared. While arguably faster, this method lacks a certain …elegance. It is fine for double-checking, but purely symbolic solutions are *always* preferred when they are possible.

The following three problems are due 23 January 2009 by the end of the day.

7. (H,R,&W 4.67) A boy whirls a stone in a horizontal circle of radius 1.5 m and at height 2.0 m above ground level. The string breaks, and the stone flies off horizontally and strikes the ground after traveling a horizontal distance of 10 m. What is the magnitude of the centripetal acceleration of the stone during the circular motion?

8. A moving particle has the position vector $\vec{\mathbf{r}}(t) = 3\cos t\,\hat{\boldsymbol{\imath}} + 4\sin t\,\hat{\boldsymbol{\jmath}}$ at time t. Find the acceleration components normal and tangential to the particle's path and the radius of curvature.

Given: the position vector of a moving particle.

Find: the normal and tangential acceleration components.

Sketch: We note that the particle is simply following an elliptical trajectory. Let $\vec{\mathbf{r}}(t) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}}$. Then $x(t) = 3\cos t$, $y(t) = 4\sin t$.

$$\frac{[x(t)]^2}{3^2} + \frac{[y(t)]^2}{4^2} = 1$$

The given trajectory is an ellipse with semimajor axis 3, and semiminor axis 4.

If you did not recognize this (and that is ok), you can always just make a nice plot:

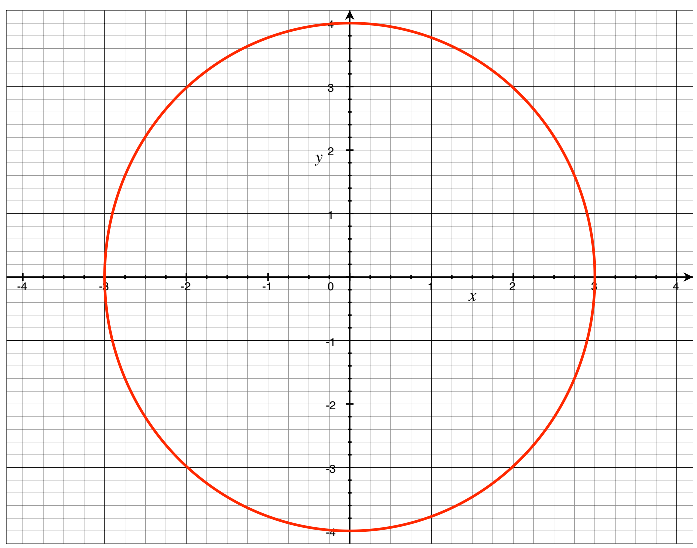


Figure 6: *The given trajectory* $\vec{\mathbf{r}}(t) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}}$.

Relevant equations: We need the definitions of the velocity and acceleration vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{a}}$, the definition of the tangential unit vector $\hat{\mathbf{T}}$ in terms of the velocity, and an equation for the magnitude of $\vec{\mathbf{a}}$ in terms of the normal and tangential components:

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}}{dt}$$

$$\vec{\mathbf{a}}(t) = \frac{d^2\vec{\mathbf{r}}}{dt^2}$$

$$\hat{\mathbf{T}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

We also need to know that the tangential component of the acceleration can be found from the scalar product of the acceleration vector and $\hat{\mathbf{T}}$:

$$a_t = \vec{\mathbf{a}} \cdot \hat{\mathbf{T}}$$

Symbolic solution: We first find the velocity and acceleration by differentiation with respect to time:

$$\begin{aligned}
\vec{\mathbf{v}}(t) &= \frac{d\vec{\mathbf{r}}}{dt} = -3\sin t\,\hat{\boldsymbol{\imath}} + 4\cos t\,\hat{\boldsymbol{\jmath}} \\
\vec{\mathbf{a}}(t) &= \frac{d^2\vec{\mathbf{r}}}{dt^2} = -3\cos t\,\hat{\boldsymbol{\imath}} - 4\sin t\,\hat{\boldsymbol{\jmath}}
\end{aligned}$$

The tangential component of the acceleration is then

$$\begin{aligned}
a_t &= \vec{\mathbf{a}} \cdot \hat{\mathbf{T}} = \vec{\mathbf{a}} \cdot \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = \frac{9\sin t\cos t - 16\sin t\cos t}{\sqrt{9\sin^2 t + 16\cos^2 t}} \\
a_t &= \frac{-7\sin t\cos t}{\sqrt{9\sin^2 t + 16\cos^2 t}} = \frac{-7\sin t\cos t}{\sqrt{9 + 7\cos^2 t}} \qquad (50)
\end{aligned}$$

The normal component is then most easily found by relating the magnitudes of the total and tangential acceleration:

$$\begin{aligned}
a_n^2 &= |\vec{\mathbf{a}}|^2 - a_t^2 = \left(9\cos^2 t + 16\sin^2 t\right) - \left[\frac{-7\sin t\cos t}{\sqrt{9\sin^2 t + 16\cos^2 t}}\right]^2 \\
a_n^2 &= 9\cos^2 t + 16\sin^2 t - \frac{(7\sin t\cos t)^2}{9\cos^2 t + 16\sin^2 t} = 9 + 7\sin^2 t + \frac{(7\sin t\cos t)^2}{9 + 7\sin^2 t} \\
\implies \quad a_n &= \sqrt{9 + 7\sin^2 t + \frac{(7\sin t\cos t)^2}{9 + 7\sin^2 t}} \qquad (51)
\end{aligned}$$

There is not much simplification to be done. Ellipses are often unfortunate in this regard.

9. Show that the curvature of a path may be determined from a particle's velocity and acceleration, *viz.*:

$$\kappa = \frac{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|}{|\vec{\mathbf{v}}|^3}$$

Recall that in terms of unit vectors tangential ($\hat{\mathbf{T}}$) and normal ($\hat{\mathbf{N}}$) to a path $s(t)$ of curvature κ, the acceleration vector is:

$$\vec{\mathbf{a}} = \frac{d^2 s}{dt^2}\,\hat{\mathbf{T}} + \kappa v^2\,\hat{\mathbf{N}}$$

Given: a proposed relationship between $\vec{\mathbf{v}}$, $\vec{\mathbf{a}}$, and κ.

Find: verify the relationship is true in general.

Sketch: N/A

Relevant equations: We need our general expression for the acceleration along a curved path in terms of the tangential and normal unit vectors $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$, the speed $|\vec{\mathbf{v}}|$, the curvature κ, and the path equation s:

$$\vec{\mathbf{a}} = \left(\frac{d^2 s}{dt^2}\,\hat{\mathbf{T}} + \kappa|\vec{\mathbf{v}}|^2\,\hat{\mathbf{N}} \right) \tag{52}$$

The only additional facts we need is the magnitude of a cross product

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin\theta_{ab} \tag{53}$$

and that by construction $\vec{\mathbf{v}}||\hat{\mathbf{T}}$, $\vec{\mathbf{v}} \perp \hat{\mathbf{N}}$.

Symbolic solution: We start by substituting our general expression for $\vec{\mathbf{a}}$ into the proposed relationship:

$$\vec{\mathbf{v}} \times \vec{\mathbf{a}} = \vec{\mathbf{v}} \times \left(\frac{d^2 s}{dt^2}\,\hat{\mathbf{T}} + \kappa|\vec{\mathbf{v}}|^2\,\hat{\mathbf{N}} \right) \tag{54}$$

The vector product $\times$ is distributive, and scalar multiplication is communtative, so this is readily rewritten as

$$\vec{\mathbf{v}} \times \vec{\mathbf{a}} = \frac{d^2 s}{dt^2}\,\vec{\mathbf{v}} \times \hat{\mathbf{T}} + \kappa|\vec{\mathbf{v}}|^2\,\vec{\mathbf{v}} \times \hat{\mathbf{N}} \tag{55}$$

By definition, $\hat{\mathbf{T}}$ points along the direction of $\vec{\mathbf{v}}$, so the two vectors are parallel. Therefore, their vector product is zero. Thus,

$$\vec{\mathbf{v}} \times \vec{\mathbf{a}} = \kappa|\vec{\mathbf{v}}|^2\,\vec{\mathbf{v}} \times \hat{\mathbf{N}} \tag{56}$$

Since we need only the *magnitude* of $\vec{\mathbf{v}} \times \vec{\mathbf{a}}$, we may write

$$|\vec{\mathbf{v}} \times \vec{\mathbf{a}}| = \kappa|\vec{\mathbf{v}}|^2\,|\vec{\mathbf{v}} \times \hat{\mathbf{N}}| = \kappa|\vec{\mathbf{v}}|^2\,|\vec{\mathbf{v}}|\,|\hat{\mathbf{N}}|\,\sin 90° = \kappa|\vec{\mathbf{v}}|^3 \tag{57}$$

because $\vec{\mathbf{v}}$ and $\hat{\mathbf{N}}$ are perpendicular by construction, and $|\hat{\mathbf{N}}|=1$ since $\hat{\mathbf{N}}$ is a unit vector. Solving our expression above for κ, we have the desired result:

$$\kappa = \frac{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|}{|\vec{\mathbf{v}}|^3} \tag{58}$$

The main advantage of this form is that it is totally coordinate-free, a topic we will return to.