UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 125 / LeClair Spring 2009

Problem Set 2

Instructions:

1. Answer all questions below. Follow the problem-solving template provided.
2. Some problems have different due dates!
3. You may collaborate, but everyone must turn in their own work

The following three problems are due 20 January 2009 at the beginning of class.

1. A rifle that shoots a bullet at $460\,\mathrm{m/s}$ is to be aimed at a target $45.7\,\mathrm{m}$ away. If the center of the target is level with the rifle, how high above the target must the rifle barrel be pointed so that the bullet hits dead center?

Given: The magnitude of the initial velocity of a fired bullet v_i and its distance from a target d.

Find: The height above the target that the shooter must aim y_{aim}. We can easily find this once we know the firing angle θ required for the bullet to hit the target. That is, the angle such that the bullet is at the same height a distance d from where it is fired.

Sketch: For convenience, let the origin be at the position the bullet is fired from. Let the $+x$ axis run horizontally, from the bullet to the target, and let the $+y$ axis run vertically. Let time $t=0$ be the moment the projectile is launched.

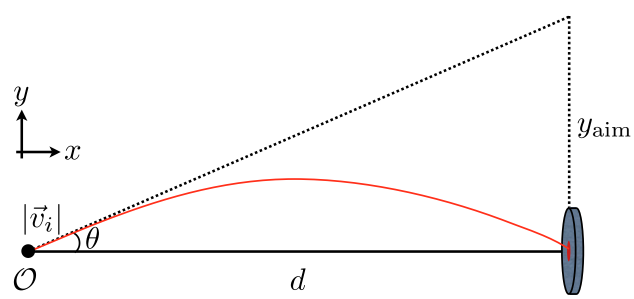


Figure 1: *Firing a rifle at a distant target. The bullet's trajectory is (approximately) shown in red.*

The bullet is fired at an initial velocity $|\vec{\mathbf{v}}_i|$ and angle θ, a distance d from a target. The target is at the same vertical position as the rifle, so we need to find the angle θ and resulting y_{aim} such that the bullet is at $y=0$ at $x=d$.

Relevant equations: In the x direction, we have constant velocity and no acceleration, with position starting at the origin at $t=0$:

$$x(t) = v_{ix}t = |\vec{\mathbf{v}}_i|\cos\theta t \tag{1}$$

In the y direction, we have an initial constant velocity of $v_{iy} = |\vec{\mathbf{v}}_i| \sin\theta$ and a constant acceleration of $a_y = -g$:

$$y(t) = v_{iy}t - \frac{1}{2}gt^2 \tag{2}$$

Solving Eq. 1 for t and substituting into Eq. 2 yields our general projectile equation, giving the path of the projectile $y(x)$ when launched from the origin with initial velocity $|\vec{\mathbf{v}}_i|$ and angle θ above the x axis:

$$y(x) = x\tan\theta - \frac{gx^2}{2|v_i|^2\cos^2\theta} \tag{3}$$

With our chosen coordinate system and origin, $y_o = 0$. We also need the aiming height above the target in terms of the target distance and firing angle, which we can get from basic trigonometry:

$$\tan\theta = \frac{y_{\text{aim}}}{d} \tag{4}$$

Note that one can also use the "range equation" directly, but this is less instructive. It is fine for you to do this in your own solutions, but keep in mind you will probably not be given these sort of specialized equations on an exam – you should know how to derive them.

Symbolic solution: We desire the bullet to reach point $(d, 0)$. Substituting these coordinates into Eq. 3, and solving for θ:

$$\begin{aligned}
y(x) &= x\tan\theta - \frac{gx^2}{2|v_i|^2\cos^2\theta} && (5)\\
0 &= d\tan\theta - \frac{gd^2}{2|v_i|^2\cos^2\theta} && (6)\\
d\tan\theta &= \frac{gd^2}{2|v_i|^2\cos^2\theta} && (7)\\
\tan\theta\cos^2\theta &= \frac{gd}{2|\vec{\mathbf{v}}_i|^2} && (8)\\
\sin\theta\cos\theta &= \frac{1}{2}\sin 2\theta = \frac{gd}{2|\vec{\mathbf{v}}_i|^2} && (9)\\
\implies \theta &= \frac{1}{2}\sin^{-1}\left[\frac{gd}{|\vec{\mathbf{v}}_i|^2}\right] && (10)
\end{aligned}$$

Given θ, rearranging Eq. 4 gives us the aiming height:

$$y_{\text{aim}} = d\tan\theta \tag{11}$$

Numeric solution: We are given $|\vec{\mathbf{v}}_i| = 460\,\text{m/s}$ and $d = 45.7\,\text{m}$:

$$\theta = \frac{1}{2}\sin^{-1}\left[\frac{gd}{|\vec{\mathbf{v}}_i|^2}\right] = \frac{1}{2}\sin^{-1}\left[\frac{\left(9.81\,\text{m/s}^2\right)(4.57\,\text{m})}{(460\text{m/s})^2}\right] \approx 0.06069° \tag{12}$$

Given the angle, we can find the height above the target we need to aim:

$$y_{\text{aim}} = d \tan\theta \approx (45.7\,\text{m}) \tan(0.06069°) \xrightarrow[\text{digits}]{\text{sign.}} 0.0484\,\text{m} = 4.84\,\text{cm} \tag{13}$$

Double check: One check is use the pre-packaged projectile range equation and make sure that we get the same answer. Given $\theta \approx 0.0607°$, we should calculate a range of d.

$$R = \frac{|\vec{\mathbf{v}}_i|^2 \sin 2\theta}{g} = \frac{(460\,\text{m/s})\,(\sin 0.1214°)}{(9.81\,\text{m/s}^2)} \approx 45.7\,\text{m} \tag{14}$$

This is not truly an independent check, since it is derived using the same equations we used above, but it is a nice indication that we haven't gone wrong anywhere.

As a more independent estimate, we can first calculate the time it would take the bullet to reach the target in the absence of gravitational acceleration – if it were just heading straight toward the target at 460 m/s. This is not so far off the real time, since the firing angle is small anyway:

$$t_{\text{est}} = \frac{d}{|\vec{\mathbf{v}}_i|} \approx 0.1\,\text{s} \tag{15}$$

In that time, how far would the bullet fall under the influence of gravity (alone)?

$$y_{\text{fall}} \approx -\frac{1}{2} g t_{\text{est}}^2 \approx 0.05\,\text{m} \tag{16}$$

Thus, we estimate that the bullet should fall about 5 cm on its way to the target, meaning we should aim about 5 cm high, in line with what we calculate by more exact means.

You can also verify that units come out correctly in Eq. 12 and Eq. 13. The argument of the $\sin^{-1}$ function must be dimensionless, as it is, and y_{aim} should come out in meters, as it does. If you carry the units through the entire calculation, or at least solve the problem symbolically, without numbers until the last step, this sort of check is trivial.

2. A particle leaves the origin with an initial velocity of $\vec{\mathbf{v}} = (3.00\,\hat{\imath})$ m/s, and a constant acceleration of $\vec{\mathbf{a}} = (-1.00\,\hat{\imath} - 0.500\,\hat{\jmath})$ m/s^2. When it reaches its maximum x coordinate, what are its velocity and position vectors?

Given: The initial velocity and acceleration vectors of a particle.

Find: The maximum x coordinate, and its velocity and position vectors at the corresponding time.

Sketch: A sketch may be a bit pedantic in this case, but here you are:

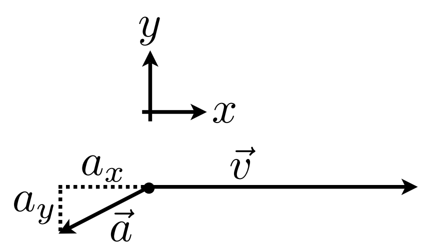


Figure 2: *A particle at the origin with an initial velocity and acceleration.*

Relevant equations: We can find the maximum x coordinate by finding the x components of the velocity and acceleration and plugging them into our derived expression for $x(t)$. This is valid because we have constant acceleration.

$$a_x = \vec{\mathbf{a}} \cdot \hat{\imath} \tag{17}$$

$$v_{ix} = \vec{\mathbf{v}} \cdot \hat{\imath} \tag{18}$$

$$x(t) = x_i + v_{ix}t + \frac{1}{2}a_x t^2 \tag{19}$$

Since our particle starts out at the origin, $x_i = 0$. Once we have $x(t)$, we can find the time at which the x coordinate is maximum, t_{max}: by differentiation:

$$\left.\frac{dx}{dt}\right|_{t_{\text{max}}} = 0 \qquad \text{and} \qquad \left.\frac{d^2x}{dt^2}\right|_{t_{\text{max}}} < 0 \tag{20}$$

Next, we need to find the position vector $\vec{\mathbf{r}}(t)$:

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t + \frac{1}{2}\vec{\mathbf{a}}\,t^2 \tag{21}$$

Again, since the particle starts at the origin, $\vec{\mathbf{r}} = 0$. Finally, we need to evaluate $\vec{\mathbf{v}}$ and $\vec{\mathbf{r}}$ at t_{max} to complete the problem.

Symbolic solution: First, we can immediately write down the equation for $x(t)$ and maximize it:

$$
\begin{aligned}
&x(t) = x_i + v_{ix}t + \frac{1}{2}a_x t^2 \\
&\left.\frac{dx}{dt}\right|_{t_{\text{max}}} = v_{ix} + a_x t_{\text{max}} = 0 \quad \Longrightarrow \quad t_{\text{max}} = -\frac{v_{ix}}{a_x} \\
&\frac{d^2x}{dt^2} = a_x
\end{aligned} \tag{22}
$$

The extreme time will correspond to a maximum position provided the second derivative is negative once we plug in the x components of the velocity and acceleration. Given the time of maximum x coordinate, we can write down the position and velocity vectors at that time readily:

$$
\begin{aligned}
&\vec{\mathbf{v}}(t_{\text{max}}) = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t_{\text{max}} = \vec{\mathbf{v}}_i - \frac{v_{ix}\vec{\mathbf{a}}}{a_x} \\
&\vec{\mathbf{r}}(t_{\text{max}}) = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t_{\text{max}} + \frac{1}{2}\vec{\mathbf{a}}\,t_{\text{max}}^2 = \vec{\mathbf{r}}_i - \frac{v_{ix}\vec{\mathbf{v}}_{\mathbf{i}}}{a_x} + \frac{v_{ix}^2\vec{\mathbf{a}}}{2a_x^2}
\end{aligned} \tag{23}
$$

Numeric solution: The x components of the velocity and acceleration are

$$
\begin{aligned}
&a_x = \vec{\mathbf{a}} \cdot \hat{\boldsymbol{\imath}} = -1.00\,\text{m/s}^2 \\
&v_x = \vec{\mathbf{v}} \cdot \hat{\boldsymbol{\imath}} = 3.00\,\text{m/s}
\end{aligned} \tag{24}
$$

which makes the time for the maximum x coordinate

$$
t_{\text{max}} = -\frac{v_{ix}}{a_x} = 3\,\text{s} \tag{25}
$$

Note also that $d^2x/dt^2 = -1.00$, which ensures that we have found a maximum of $x(t)$. At this point, we can just write down the expressions for $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{v}}(t)$ and plug in the numbers:

$$
\begin{aligned}
\vec{\mathbf{v}}(t_{\text{max}}) &= \vec{\mathbf{v}}_i + \vec{\mathbf{a}}\,t_{\text{max}} \\
&= (3.00\,\text{m/s}\,\hat{\boldsymbol{\imath}}) + \left(-1.00\,\text{m/s}^2\,\hat{\boldsymbol{\imath}} - 0.500\,\text{m/s}^2\,\hat{\boldsymbol{\jmath}}\right)(3.00\,\text{s}) = (-1.5\,\hat{\boldsymbol{\jmath}})\ \text{m/s}
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\vec{\mathbf{r}}(t_{\text{max}}) &= \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t_{\text{max}} + \frac{1}{2}\vec{\mathbf{a}}\,t_{\text{max}}^2 \\
&= 0 + (3.00\,\text{m/s}\,\hat{\boldsymbol{\imath}})(3.00\,\text{s}) + \frac{1}{2}\left(-1.00\,\text{m/s}^2\,\hat{\boldsymbol{\imath}} - 0.500\,\text{m/s}^2\,\hat{\boldsymbol{\jmath}}\right)(3.00\,\text{s})^2 \\
&= (9.00\,\text{m}\,\hat{\boldsymbol{\imath}}) + (-4.50\,\text{m}\,\hat{\boldsymbol{\imath}} - 2.25\,\text{m}\,\hat{\boldsymbol{\jmath}}) = (4.5\,\hat{\boldsymbol{\imath}} - 2.25\,\hat{\boldsymbol{\jmath}})\ \text{m}
\end{aligned} \tag{27}
$$

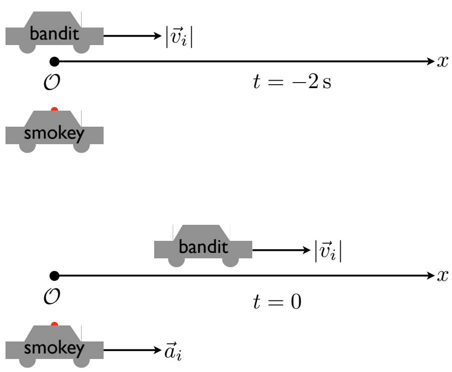
Double check: We carried the units throughout our calculations, and can be fairly confident that they are correct. We also performed the second derivative test to ensure that we found a maximum in $x(t)$.

3. A car is traveling at a constant velocity of $18\,\mathrm{m/s}$ and passes a police cruiser. Exactly $2.0\,\mathrm{s}$ after passing, the cruiser begins pursuit, with a constant acceleration of $2.5\,\mathrm{m/s^2}$. How long does it take for the cruiser to overtake the car (from the moment the cop car starts)?

Given: The constant velocity and initial position of a speeding car, the constant acceleration and initial position of a pursuing police cruiser $2.0\,\mathrm{s}$ later.

Find: How long it takes the police cruiser to overtake the car. This means we need the position versus time for each vehicle, from which we can find their intersection point.

Sketch: Let the car's and cruiser's initial position be $x=0$, with the direction of travel being the $+x$ axis. Let time $t=0$ be when *the cruiser begins accelerating*. Thus, the car starts off with $\vec{\mathbf{v}}_i=18\,\mathrm{m/s}\,\hat{\imath}$ at $t=-2\,\mathrm{s}$ from $x=0$, and the police cruiser starts off with $\vec{\mathbf{a}}_i=2.5\,\mathrm{m/s^2}\,\hat{\imath}$ at $t=0$, also from $x=0$.



A car chase. **Upper:** *at time $t=-2\,s$ the car passes the stationary police cruiser.* **Lower** *at time $t=0\,s$, the police cruiser begins chase with constant acceleration.*

Relevant equations: We need the general equation for position in one dimension under the conditions of constant (or zero) acceleration:

$$x(t) = x_i + v_{ix}t + \frac{1}{2}a_x t^2 \tag{28}$$

Symbolic solution: For the car, we have $x_i=0$ and $a_x=0$. Let the car's initial velocity be v_{ix}. The car has been moving at constant velocity for two seconds before the cruiser starts, so we need to shift the time coordinate from $t \mapsto (t+2)$.

$$x_{\text{car}} = v_{ix}\,(t + 2\,\mathrm{s}) \tag{29}$$

You can verify that this correctly gives $x_{\text{car}}\,(-2\,\mathrm{s})=0$. In order to keep things more general, however, let us say that the cruiser starts δt seconds later (we can later set $\delta t=2\,\mathrm{s}$):

$$x_{\text{car}} = v_{ix}\,(t + \delta t) \tag{30}$$

For the police cruiser, we have $x_i=0$ and $v_{ix}=0$. Let the cruiser's initial acceleration be a_x. The cruiser starts out at $t=0$, so things are simple:

$$x_{\text{cop}} = \frac{1}{2}a_x t^2 \tag{31}$$

We need to find the time t at which $x_{\text{car}} = x_{\text{cop}}$, from which we can get the time it takes for the police cruiser to overtake the car, *viz.* $t - 2$.

$$\begin{aligned}
x_{\text{car}} &= x_{\text{cop}} && (32)\\
v_{ix}\left(t+\delta t\right) &= \frac{1}{2}a_x t^2 && (33)\\
v_{ix}t + \delta t\, v_{ix} &= \frac{1}{2}a_x t^2 && (34)\\
0 &= a_x t^2 - 2v_{ix}t - 2\delta t\, v_{ix} && (35)\\
\implies \quad t &= \frac{2v_{ix} \pm \sqrt{4v_{ix}^2 + 8\delta t\, v_{ix} a_x}}{2a_x} && (36)
\end{aligned}$$

It is the "+" solution in the $\pm$ we want, as our numerical solution below will verify.

Numeric solution: Using the numbers we are given:

$$\begin{aligned}
t &= \frac{2v_{ix} \pm \sqrt{4v_{ix}^2 + 8\delta t\, v_{ix} a_x}}{2a_x} = \frac{2\left(18\,\text{m/s}\right) \pm \sqrt{4\left(18\,\text{m/s}\right)^2 + 8\left(2\,\text{s}\right)\left(18\,\text{m/s}\right)\left(2.5\,\text{m/s}^2\right)}}{2\left(2.5\,\text{m/s}^2\right)}\\
&= \frac{36\,\text{m/s} \pm \sqrt{1296\,\text{m}^2/\text{s}^2 + 720\,\text{m}^2/\text{s}^2}}{5\,\text{m/s}^2} = \frac{\left(36 \pm 44.9\right)\ \text{m/s}}{5\,\text{m/s}^2} = \{16.2, -1.78\}\ \text{s} && (37)
\end{aligned}$$

The negative solution we can reject as unphysical, so it takes approximately 16 s for the cruiser to overtake the car.

Double check: We carried units throughout the numerical phase of the calculation, and our end result comes out in seconds as required. Another check we can make is to calculate the position of both the car and cruiser at the time we found above to be sure they are the same:

$$\begin{aligned}
x_{\text{car}} &= v_{ix}\left(t+\delta t\right) = \left(18\,\text{m/s}\right)\left(18.2\,\text{s}\right) \approx 327.6\,\text{m} \xrightarrow[\text{digits}]{\text{sign.}} 330\,\text{m} && (38)\\
x_{\text{cop}} &= \frac{1}{2}a_x t^2 = \frac{1}{2}\left(2.5\,\text{m/s}^2\right)\left(16.2\,\text{s}\right)^2 \approx 328.05\,\text{m} \xrightarrow[\text{digits}]{\text{sign.}} 330\,\text{m} && (39)
\end{aligned}$$

Within the precision implied by the number of significant digits, the two positions are the same; our answer is reasonable.

The following three problems are due 22 January 2009 at the beginning of class.

4. A projectile is launched with initial velocity $\vec{\mathbf{v}}_i$ and angle θ a distance d from a ramp inclined at angle φ (see figure below). What is the constraint on the initial velocity and angle for the projectile to hit the ramp (*i.e.*, it does not fall short)? *No numerical solution is required.*

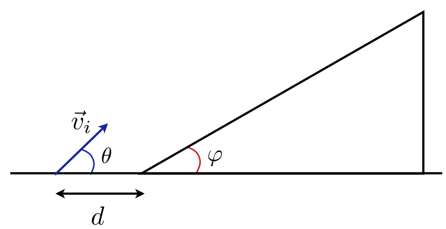


Figure 3: *Problems 4 and 5: a projectile is launched with initial velocity* $\vec{\mathbf{v}}_i$ *and angle* θ *a distance* d *from a ramp inclined at angle* φ.

5. Referring to the preceding problem, how far along the ramp (laterally), and at what height, does the projectile hit the ramp? You may assume that the ramp is incredibly long. *No numerical solution is required.*

6. A ball rolls horizontally off the top of a stairway with a speed of 1.52 m/s. The steps are 20.3 cm high and 20.3 cm wide. Which step does the ball hit first. You may assume that there are many, many stairs.

The following three problems are due 23 January 2009 by the end of the day.

7. A boy whirls a stone in a horizontal circle of radius 1.5 m and at height 2.0 m above ground level. The string breaks, and the stone flies off horizontally and strikes the ground after traveling a horizontal distance of 10 m. What is the magnitude of the centripetal acceleration of the stone during the circular motion?

8. A moving particle has the position vector $\vec{\mathbf{r}}(t) = 3\cos t\,\hat{\imath} + 4\sin t\,\hat{\jmath}$ at time t. Find the acceleration components normal and tangential to the particle's path and the radius of curvature.

9. Show that the curvature of a path may be determined from a particle's velocity and acceleration, *viz.*:

$$\kappa = \frac{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|}{|\vec{\mathbf{v}}|^3}$$

Recall that in terms of unit vectors tangential ($\hat{\mathbf{T}}$) and normal ($\hat{\mathbf{N}}$) to a path $s(t)$ of curvature κ, the acceleration vector is:

$$\vec{\mathbf{a}} = \frac{d^2 s}{dt^2}\,\hat{\mathbf{T}} + \kappa v^2\,\hat{\mathbf{N}}$$