UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 125 / LeClair Spring 2009

Problem Set 3: Solutions

The following problems are due 27 January 2009 at the beginning of class.

1. A person standing at the top of a hemispherical rock of radius R kicks a ball (initially at rest on the top of the rock) to give it horizontal velocity $\vec{\mathbf{v}}_i$.

(a) What must be its minimum initial speed if the ball is never to hit the rock after it is kicked?

(b) With this initial speed, how far from the base of the rock does the ball hit the ground?

Find: The minimum speed for the ball not to hit the rock, and the net horizontal distance at that speed. Since the rock may be described by a circle, and the ball's motion a parabola, we are seeking a condition on the initial velocity such that the parabola always lies above the circle.

Given: The geometry of the rock, the ball's initial velocity.

Sketch: Let the x axis run horizontally and the y axis vertically, with the origin at the rock's center. This makes the ball's starting position $(0, R)$ and its launch angle with respect to the x axis $\theta = 0$.

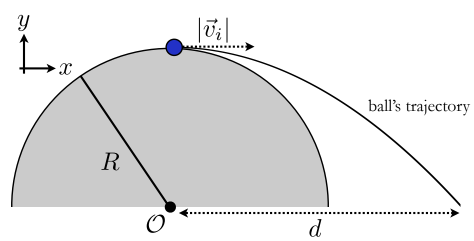

Figure 1: *A ball is kicked off the top of a rock by an unseen person.*

Relevant equations: We need the equation of a circle of radius R centered on the origin, and the trajectory of a projectile fired at angle $\theta = 0$ relative to the x axis with starting vertical position $y(0) = R$. Let the circle be described by $y_p(x)$ and the rock $y_r(x)$. Since our solution is restricted to the upper right quadrant, the rock may be described by

$$y_r(x) = \sqrt{R^2 - x^2} \tag{1}$$

The ball's trajectory is our well-known result

$$y_p(x) = y(0) + (\tan\theta)\,x - \frac{gx^2}{2|\vec{\mathbf{v}}|^2\cos^2\theta} = R - \frac{gx^2}{2|\vec{\mathbf{v}}|^2} \tag{2}$$

Symbolic solution: The condition that the ball does not hit the rock is simply that the parabola and circle above have no intersection point, other than the trivial one at $(0, R)$. That is, the parabola must lie above the circle everywhere except $(0, R)$. Thus,

$$
\begin{aligned}
y_p(x) &\geq y_r(x) \\
R - \frac{gx^2}{2|\vec{\mathbf{v}}|^2} &\geq \sqrt{R^2 - x^2} \qquad (3)
\end{aligned}
$$

In principle, this is it. Much algebra now ensues. First, simply square both sides and simplify. Since both sides must be positive for all x considered, by the problem's construction, this does not alter the inequality.

$$
\begin{aligned}
\left(R - \frac{gx^2}{2|\vec{\mathbf{v}}|^2}\right)^2 &\geq \left(\sqrt{R^2 - x^2}\right)^2 \\
R^2 - \frac{gRx^2}{|\vec{\mathbf{v}}|^2} + \frac{g^2x^4}{4|\vec{\mathbf{v}}|^4} &\geq R^2 - x^2 \\
\left(\frac{g^2}{4|\vec{\mathbf{v}}|^2}\right)x^4 + \left(1 - \frac{gR}{|\vec{\mathbf{v}}|^2}\right)x^2 &\geq 0 \\
x^2\left(\frac{g^2}{4|\vec{\mathbf{v}}|^2}x^2 + 1 - \frac{gR}{|\vec{\mathbf{v}}|^2}\right) &\geq 0 \\
x^2\left(\frac{g^2}{4|\vec{\mathbf{v}}|^2}x^2 + 1 - \frac{gR}{|\vec{\mathbf{v}}|^2}\right) &\geq 0 \qquad (x \neq 0) \\
\frac{g^2}{4|\vec{\mathbf{v}}|^2}x^2 + 1 - \frac{gR}{|\vec{\mathbf{v}}|^2} &\geq 0 \\
\frac{g}{4|\vec{\mathbf{v}}|^2}x^2 &\geq \left(\frac{gR}{|\vec{\mathbf{v}}|^2} - 1\right) \qquad (4)
\end{aligned}
$$

We require this inequality to be true for all $x > 0$ for the ball not to hit the rock anywhere in the domain of interest. The only way this can happen is if the right-hand side is negative:

$$
\begin{aligned}
&\frac{gR}{|\vec{\mathbf{v}}|^2} - 1 \leq 0 \\
\Longrightarrow \quad &|\vec{\mathbf{v}}| \geq \sqrt{gR} \qquad (5)
\end{aligned}
$$

Note that this is *not* the same condition you would find by simply requiring the particle's *range* to be larger than R. It is easy to verify that one can make a parabola with horizontal range R in this situation that still intersects the circle ... try it out!

Where does the projectile land? Clearly, at $y_p = 0$, since that is where the ground is. We simply need to set the ball's y position equal to zero, and solve for the resulting x value using our minimal velocity from Eq. 5. This is where the ball lands.

$$
\begin{aligned}
y_p = 0 = R - \frac{gx^2}{2|\vec{\mathbf{v}}|^2} &= R - \frac{gx^2}{2gR} = R - \frac{x^2}{2R} \\
2R^2 &= x^2 \\
\Longrightarrow \quad x &= R\sqrt{2} \qquad (6)
\end{aligned}
$$

We are asked *how far from the rock* the ball lands. Since the rock extends to $x = R$, we have gone beyond that by a distance

$$
\text{distance from rock} = R\left(\sqrt{2} - 1\right) \qquad (7)
$$

Numeric solution: Numbers? How awkward. $\sqrt{2} \approx 1.41$, $\sqrt{2}-1 \approx 0.41$. The ball lands about 40% of the rock's radius beyond its base. With $g \approx 10$, and $\sqrt{g} \approx 3.2$, the maximal velocity is about $3.2\sqrt{R}$.

Double check: Things you can do: simply graph the two trajectories you came up with for a given value of R, and verify they do not intersect. Check the units of the final answer. Check that the ball lands beyond the base of the rock (it does).

Another way: Since the parabola has a maximal radius of curvature at at its apex, with a little geometrical reasoning you can prove that if the circle and parabola are tangent at the parabola's apex, and the parabola's radius of curvature there exceeds R, the two curves cannot intersect. It does work: calculate the parabola's radius of curvature, insist that it be larger than R, and the same condition results: $v \geq \sqrt{gR}$. I don't really have the stamina to work up a full geometric proof of that, however …perhaps one of you would do it for extra credit?

2. The air resistance experienced by an object in free fall can be modeled as an additional force opposite the direction of motion proportional to velocity squared:[i]

$$\vec{\mathbf{F}}_{\text{drag}} = -\frac{1}{2} D \rho A |\vec{\mathbf{v}}| \vec{\mathbf{v}}$$

where D is an empirical constant, ρ the density of air, A the cross sectional area of the falling body, and v its velocity. For an object in free-fall, this drag force counteracts the gravitational force $F_g = mg$. Find an expression for the terminal velocity of a falling body, assuming the object is traveling straight downward.

Find: The terminal velocity of a body which experiences a gravitational and drag force. Terminal velocity means that the velocity is constant, which means zero acceleration, which means no net force.

Given: An equation for drag force in terms of a body's speed, cross-sectional area, and numerical constants. Technically, the area should be the orthographic projection of the body, or the area projected on a plane perpendicular to the direction of motion. Since this is an approximate law, with fudge-factor D already included … we get the feeling it is not worth squabbling over.

Sketch: Watch the end of *Point Break*,[ii] you should get the gist of it.

Relevant equations: We are given the equation for drag force and gravitational force. We need additionally only Newton's second law:

$$\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}} \tag{8}$$

Symbolic solution: The gravitational force clearly acts in the same direction as the velocity, since it causes the free fall velocity in the first place. The drag force acts in the opposite direction, by the definition given. At constant terminal velocity v_t, acceleration must be zero and so must the force balance. Since we have only two forces acting in opposite directions, their magnitudes must be equal:

[i] See `http://en.wikipedia.org/wiki/Drag_(physics)`, which actually has a solution to this very problem.
[ii] `http://www.imdb.com/title/tt0102685/`

$$\sum \vec{\mathbf{F}} = \vec{\mathbf{F}}_g - \vec{\mathbf{F}}_{\text{drag}} = 0$$

$$\implies |\vec{\mathbf{F}}_g| = |\vec{\mathbf{F}}_{\text{drag}}| \tag{9}$$

$$mg = \left|\frac{1}{2}D\rho A|\vec{\mathbf{v}}|\vec{\mathbf{v}}\right| = \frac{1}{2}D\rho A|\vec{\mathbf{v}}|^2$$

$$\implies |\vec{\mathbf{v}}| = \sqrt{\frac{2mg}{D\rho A}} \tag{10}$$

Numeric solution: n/a

Double check: What is sensible? The more cross-sectional area a body has, the slower it is at terminal velocity. Conversely, smaller cross-sectional areas provide higher terminal velocities, which is how Keanu Reeves managed to catch Patrick Swayze in mid-air at the end of *Point Break*. This effect should also be enhanced when falling through denser air (larger ρ). Both effects are borne out by our equation above.

3. A ball is dropped from a window 10 m above the ground at $t=0$. When it bounces, its rebound speed is 71% of its impact speed. At $t=2$ s, a second ball is released from the same place. Take $g=10\,\text{m/s}^2$.

(a) Give an expression for the position of the first ball $y_1(t)$, that is valid after it has bounced once.

(b) When will the two balls collide? After how many total bounces?

Given: The height at which two balls are dropped, the time interval between dropping them, and the relative rebound speed of the first ball.

Find: The time at which the two balls hit each other, and after how many bounces this occurs.

"Sketch:" We need only one dimension, since this is free fall motion along a straight line. Let this one dimension be an x axis running vertically, with $+x$ in the upward direction. Let the origin be at the ground level, making the initial height of both balls $x_o = 10$ m. The first ball is dropped at $t=0$ and bounces the first time at t_b, while the second ball is released at time $t = t_2$. After the first ball bounces, its rebound velocity is a factor $\alpha = 0.71$ smaller than just before it bounces.

Relevant equations: We need only the one-dimensional equation of motion with constant acceleration a:

$$x(t) = x_0 + v_{ix}t + \frac{1}{2}at^2 \tag{11}$$

Symbolic solution: Worry about the first ball first. It leaves at time $t=0$, with $v_{ix}=0$ and $x_o = 10$ m. Its motion, up until it hits the ground, is described by:

$$x_{1,1} = x_o - \frac{1}{2}gt^2 \tag{12}$$

Here the subscript "1,1" means the first ball, first bounce. The first bounce occurs at time t_b when $x=0$:

$$0 = x_o - \frac{1}{2}gt_b^2 \qquad \implies \qquad t_b = \sqrt{\frac{2x_o}{g}} \tag{13}$$

At the moment it strikes the ground, the first ball's velocity is

$$v_{1,1} = \left.\frac{dx_{1,1}}{dt}\right|_{t_b} = -gt_b = -\sqrt{2x_o g} \tag{14}$$

At the moment it rebounds, the first ball has a new velocity $v_{1,2}$ which is a factor α less than its velocity just before the rebound (viz., $v_{1,1}$):

$$v_{1,2} = \alpha|v_{1,1}| = \alpha\sqrt{2x_o g} \tag{15}$$

Given this velocity and the fact that the first ball now starts from $x=0$, we can immediately write down the equation of motion for the first ball between the first and second bounces. *However:* we have to now take into account the fact that the time coming into the equation of motion has to be *the time after the first bounce*, not just "t" which is the time since the beginning of the problem. This means that instead of t in our equation, we need $t - t_b$ – whatever the time is, we have to subtract off the first t_b seconds that happened before the bounce.

$$x_{1,2}(t) = v_{1,2}\left(t - t_b\right) - \frac{1}{2}g\left(t - t_b\right)^2 \tag{16}$$

How do you know to put a "+" or "-"? The easiest way is to check your boundary conditions – at time t_b, we must have $x_{1,2}=0$, that is the simplest way. The second bounce for the first ball occurs when we set *this* equation to zero:

$$0 = v_{1,2}\left[t - t_b\right] - \frac{1}{2}g\left[t - t_b\right]^2 \tag{17}$$

Before you go any further, let $\Delta t = t - t_b$.

$$\begin{aligned} 0 &= v_{1,2}\Delta t - \frac{1}{2}g\left(\Delta t\right)^2 \\ 0 &= \Delta t\left(v_{1,2} - \frac{1}{2}g\Delta t\right) \end{aligned} \tag{18}$$

$$\implies \quad \Delta t = \frac{2v_{1,2}}{g} = t - t_b \tag{19}$$

$$t = t_b + \frac{2v_{1,2}}{g} = t_b + \frac{2\alpha|v_{1,1}|}{g} = t_b + \frac{2\alpha\sqrt{2x_o}}{\sqrt{g}} = t_b + 2\alpha t_b = \left(1 + 2\alpha\right)t_b \tag{20}$$

So long as the second ball reaches the ground before this time, but after t_b the balls must hit between the first and second bounces of the first ball. Let us see how that works out.

Now, what about the second ball? It is released at a time t_2, so its position is described by

$$x_{2,1}(t) = x_o - \frac{1}{2}g\left(t - t_2\right)^2 \tag{21}$$

Just like before, we have to subtract off the t_2 seconds of delay to account for when we started our "clock" at $t=0$. Again, you can check $x_{2,1}(t_2)=0$ to be sure you have it the right way. The time it takes to bounce the first time is precisely the same as the first ball, so it happens at a time

$$t = t_b + t_2 \qquad \text{for first bounce of second ball} \tag{22}$$

This is the latest possible time the two balls can collide, if you think about it ... Our condition that the two balls should collide between the first and second bounce of the first ball would then read

$$(1+2\alpha)\,t_b \geq t_b + t_2 \tag{23}$$
$$2\alpha t_b \geq t_2 \tag{24}$$

If this is satisfied, we know the balls hit between the first and second bounce of the first ball. Checking with the numbers given, we find that the inequality is essentially *exact*. That is, we should find that the two balls hit the ground together, as the first ball bounces a second time, and the second bounces a first time.

To prove this, we simply have to set the equation of motion for the first ball after its first bounce equal to that of the second ball and solve for t. Provided that we find $x > 0$ for either ball at that time, our solution is valid.

$$x_{2,1}(t) = x_o - \frac{1}{2}g\,(t-t_2)^2 = x_{1,2}(t) = v_{1,2}\,(t-t_b) - \frac{1}{2}g\,(t-t_b)^2$$
$$x_o - \frac{1}{2}gt^2 + gt_2t - \frac{1}{2}gt_2^2 = v_{1,2}t - v_{1,2}t_b - \frac{1}{2}gt^2 + gt_bt - \frac{1}{2}gt_b^2 \tag{25}$$
$$x_o + gt_2t - \frac{1}{2}gt_2^2 = v_{1,2}t - v_{1,2}t_b + gt_bt - \frac{1}{2}gt_b^2 \tag{26}$$
$$(v_{1,2} - gt_2 + gt_b)\,t = x_o + v_{1,2}t_b - \frac{1}{2}gt_2^2 + \frac{1}{2}gt_b^2 \tag{27}$$
$$t = \frac{x_o + v_{1,2}t_b + \frac{1}{2}g\left(t_b^2 - t_2^2\right)}{v_{1,2} + g\,(t_b - t_2)} \tag{28}$$

It is useful to notice at this point that $\frac{1}{2}gt_b^2 = x_o$, $v_{1,2}t_b = 2\alpha x_o$, and $gt_b = \sqrt{2x_og}$ to put our expression in terms of known quantities.

$$t = \frac{x_o + 2\alpha x_o + x_o - \frac{1}{2}gt_2^2}{v_{1,2} + g\,(t_b - t_2)} = \frac{(1+\alpha)\,t_b^2 - \frac{1}{2}t_2^2}{(1+\alpha)\,t_b - t_2} \tag{29}$$

You need not be as obsessive as I am with reducing formulas, but the result is much nicer than it has any right to be.

Numeric solution: With the number given, we find the collision at $t \approx 3.4\,\mathrm{s}$, just at the end of the second bounce for the first ball.

Double check: The units are correct everywhere. You can verify that the collision time is indeed when the first ball bounces the second time, at $(1+2\alpha)\,t_b \approx 3.4\,\mathrm{s}$, and the position of both balls is $x \approx 0$.

The following problems are due 29 January 2009 at the beginning of class.

4. Exam review: A block of mass m is released from rest at h above the surface of a table, at the top of an incline of inclination θ, as shown below. The *frictionless* incline is fixed on a table of height H.

(a) Determine the acceleration of the block as it slides down the incline.

(b) What is the velocity of the block as it leaves the incline?

(c) How far from the table will the block hit the floor?

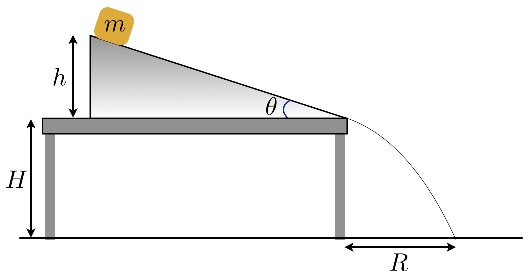

Figure 2: *A block is let go from the top of a ramp sitting on a table.*

5. Problem 6.55 from your textbook.

6. Problem 6.59 from your textbook.

The following three problems are due 30 January 2009 by the end of the day.

7. Problem 6.34 from your textbook.

8. Problem 6.60 from your textbook.

9. A mass m is released from rest at height h on the top of a ramp of inclination θ. The coefficient of kinetic friction between the ramp and mass is μ_k. The block slides down the ramp, up a second identical ramp, back down again, and so forth.

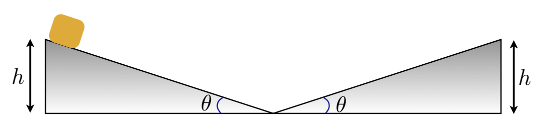

Figure 3: *A block is let go from the top of a ramp sitting on a table.*

(a) After one round trip (from the top of the first ramp and back again), how far away from the mass from its starting point?

(b) How about after n round trips?

(c) Does the mass ever stop?