UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 125 / LeClair Spring 2009

Problem Set 4: Solutions

The following problems are due 5 February 2009 at the beginning of class.

1. One of the questions you did *not* solve for the exam.

See the separate exam solutions.

2. Problem 7.10 from your textbook

Find: the amount of work done by a constant force through a given displacement.

Given: the force and displacement vectors.

Symbolic solution: Given a *constant* force and displacement, the work is simply the scalar product of the two. In two dimensions,

$$\begin{aligned} \text{let} \quad & \vec{\mathbf{F}} = F_x\,\hat{\imath} + F_y\,\hat{\jmath} \\ \text{let} \quad & \vec{\mathbf{d}} = d_x\,\hat{\imath} + d_y\,\hat{\jmath} \\ \Longrightarrow \quad & W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = F_x d_x + F_y d_y \end{aligned}$$

Numeric solution: We are given $\vec{\mathbf{d}} = 15\,\hat{\imath} - 12\,\hat{\jmath}$ and $\vec{\mathbf{F}} = 210\,\hat{\imath} - 150\,\hat{\jmath}$. Thus,

$$W = (15\,\mathrm{m})\,(120\,\mathrm{N}) + (-12\,\mathrm{m})\,(-150\,\mathrm{N}) = 4.95 \times 10^3\,\mathrm{J}$$

Here we made use of the fact that $1\,\mathrm{N} \cdot 1\,\mathrm{m} = 1\,\mathrm{J}$.

The following problems are due 6 February 2009 at the end of the day.

3. A second one of the questions you did *not* solve for the exam.

See the separate exam solutions.

4. Problem 7.33 from your textbook

Find: the distance a block pushed with a constant force will compress a spring, the amount of work done, and the position where the kinetic energy is maximum.

Given: The magnitude of the constant force F and the spring's force constant k.

Symbolic solution: The block will stop when the amount of work done by the applied force equals the amount of work done in compressing the spring – after that point, no further compression is possible. We have already derived the work done in compressing a spring by a distance x. Since the applied force is constant, the work done by it is simply its magnitude times the displacement. Let x_{stop} be the distance the block compresses the spring before stopping:

$$\begin{aligned}W_s &= W_f \\ \frac{1}{2}kx_{\text{stop}}^2 &= Fx_{\text{stop}} \\ x_{\text{stop}} &= \frac{2F}{k}\end{aligned}$$

Note that this is *not* the point at which the applied force balances the spring force (which would be $x=F/k$). This only gives the point of zero acceleration, not the stopping point, the point of zero *velocity*. Knowing the stopping distance, the work done by the applied force on the block is easily found:

$$W_f = Fx_{\text{stop}} = \frac{2F^2}{k}$$

The work done by the spring force must be the same, but we can calculate it easily in any case:

$$W_s = \frac{1}{2}kx_{\text{stop}}^2 = \frac{1}{2}k\left(\frac{2F}{k}\right)^2 = \frac{2F^2}{k}$$

Finally, we require the position at which the block has maximum kinetic energy. At any point during the block's displacement, the change in kinetic energy compared to the starting point is given by the net work. The block starts at rest, and the net work is the difference between the work done by the constant force on the block and the work the block does in compressing the spring.

$$\Delta K = K_f - K_i = K_f = W_f - W_s = Fx - \frac{1}{2}kx^2$$

We can extremize the kinetic energy by taking its derivative with respect to position:

$$\frac{dK_f}{dx} = F - kx = 0 \qquad \Longrightarrow \qquad x_{\text{max K}} = \frac{F}{k}$$

The second derivative is negative for all x, so we have found a maximum. Not surprisingly, finding the maximal kinetic energy is equivalent to finding the point of zero net force. This makes some sense – the maximum kinetic energy should occur when the spring force, which serves only to retard the block's motion, is perfectly balanced by the external force. The maximum kinetic energy is then

$$K_{\text{max}} = Fx_{\text{max K}} - \frac{1}{2}kx_{\text{max K}}^2 = \frac{F^2}{k} - \frac{1}{2}k\left(\frac{F}{k}\right)^2 = \frac{F^2}{2k}$$

Kinetic energy at best accounts for one quarter of the the work supplied by the constant force.

Numeric solution: With the numbers given, $k=50\,\text{N/m}$ and $F=3.0\,\text{N}$,

$$x_{\text{stop}} = \frac{2F}{k} = \frac{6.0}{50}\,\text{m} = 0.12\,\text{m}$$

$$W_f = W_s = \frac{2F^2}{k} = \frac{2\left(9\,\mathrm{N}^2\right)}{50\,\mathrm{N/m}} = 0.36\,\mathrm{J}$$

$$x_{\text{max K}} = \frac{F}{k} = \frac{3.0\,\mathrm{N}}{50\,\mathrm{N/m}} = 0.06\,\mathrm{m}$$

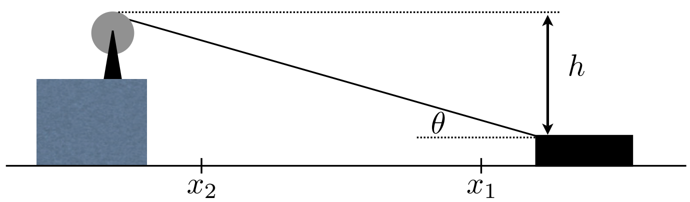
$$K_{\text{max}} = \frac{F^2}{2k} = \frac{9\,\mathrm{N}^2}{100\,\mathrm{N/m}} = 0.09\,\mathrm{J}$$

5. Problem 7.42 from your textbook

Find: The change in kinetic energy of a block pulled on a frictionless surface from x_1 to x_2 by a string with tension T a height h above the block.

Given: The string's tension, the geometry of the system.

Sketch:



Problem 7.42

Symbolic solution: There are two straightforward ways to do this problem. First, the more formal. The work done by the tension force of the rope must be equal to the block's change in kinetic energy. At any point during the block's motion, the angle at which the rope is pulling changes, which means that the horizontal force on the block changes. Let the x axis be horizontal, with the $+x$ direction running to the right, and the y axis be vertical with the $+y$ direction being upward. Let the origin be at the base of the platform holding the pulley.

At any horizontal distance x, we can resolve the rope's tension along the x and y axes using the geometry of the system:

$$\vec{\mathbf{T}} = -T\cos\theta\,\hat{\boldsymbol{\imath}} + T\sin\theta\,\hat{\boldsymbol{\jmath}} = \frac{-Tx}{\sqrt{x^2+h^2}}\,\hat{\boldsymbol{\imath}} + \frac{Th}{\sqrt{x^2+h^2}}\,\hat{\boldsymbol{\jmath}}$$

The block moves purely along the x axis, so an incremental displacement of the block can be written $d\vec{\mathbf{x}}=dx\,\hat{\boldsymbol{\imath}}$. The work done by the rope's tension is the integral of $\vec{\mathbf{T}}\cdot d\vec{\mathbf{x}}$ along the block's path:

$$W = \int_{x_1}^{x_2} \vec{\mathbf{T}}\cdot d\vec{\mathbf{x}} = \int_{x_1}^{x_2} \frac{-Tx}{\sqrt{x^2+h^2}}\,dx = -T\sqrt{x^2+h^2}\Big|_{x_1}^{x_2} = T\sqrt{h^2+x_1^2} - T\sqrt{h^2+x_2^2}$$

The work done by the tension must be equal to the change in the block's kinetic energy.

$$W = K_f - K_i = T\sqrt{h^2 + x_1^2} - T\sqrt{h^2 + x_2^2}$$

The second method is to recognize how much work is done in creating the initial and final situations. At the beginning, we have a certain length of rope l_i with a certain tension T applied. It takes $W_i = Tl_i$ worth of work to apply tension T to a length l_i of rope. At the end of the block's motion, we have a shorter length of rope l_f with the same tension applied. The difference in work required to tension the rope from start to finish must be the work applied to the block, and this difference must be gained by the block as kinetic energy. Simple geometry gives us the starting and ending lengths of rope:

$$K_f - K_i = Tl_f - Tl_i = T\sqrt{h^2 + x_1^2} - T\sqrt{h^2 + x_2^2}$$

Numeric solution: Given $T = 25.0\,\mathrm{N}$, $h = 1.20\,\mathrm{m}$, $x_1 = 3.00$ and $x_2 = 1.00$

$$W = \Delta K = T\left[\sqrt{h^2 + x_1^2} - \sqrt{h^2 + x_2^2}\right] = (25.0\,\mathrm{N})\left[\sqrt{1.20^2 + 3.00^2}\,\mathrm{m} - \sqrt{1.20^2 + 1.00^2}\,\mathrm{m}\right] \approx 41.7\,\mathrm{J}$$