UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

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Problem Set 6: Solutions

Solutions not yet completed: Halliday, Resnick, & Walker, problems **9.48, 9.53, 9.59, and 9.68**

1. Halliday, Resnick, & Walker, problem **9.5**

The key to solving this problem easily is to recognize that we can handle the system in parts. We can first find the center of mass of the three hydrogen atoms, and replace all three by a single equivalent mass at the center of mass position. Then we can use that equivalent mass and the nitrogen atom to find the total center of mass.

The three hydrogen atoms (each of mass m_h) form an equilateral triangle, and by symmetry their center of mass must be at the center of the triangle. Conveniently, the center of the triangle formed by the three hydrogen atoms has been placed at the origin. That means we can replace the three hydrogen atoms by a single equivalent mass $3m_h$ at the origin, independent of the presence of the nitrogen atom.

Now we need to find the center of mass of the equivalent mass $3m_h$ at the origin and the nitrogen atom (mass m_n) at position $y_n = \sqrt{L^2 - d^2}$. By symmetry, the center of mass must be along the y axis, so $x_{com} = z_{com} = 0$. From the definition of y_{com}, and noting $m_n/m_h \approx 13.9$,

$$y_{com} = \frac{1}{3m_h + m_n}\left(\sqrt{L^2 - d^2} + 0\right) = \frac{m_n/m_h}{3 + m_n/m_h}\sqrt{L^2 - d^2} \approx 3.13 \times 10^{-11}\text{ m}$$

What about doing the problem without using symmetry, and simply writing down the (x, y, z) coordinates of each atom? It works out the same way, even if you don't first find the COM of the three hydrogen atoms separately. We will leave to you to verify this.

2. Halliday, Resnick, & Walker, problem **9.8**

Let the x axis run vertically, with its origin at the base of the can and $+x$ being upward. Let the height of the can be h.

We can first model the soda and the can itself as point particles at their respective centers of mass. The can is easy: it has a mass m_c and its center of mass must be halfway up the can at $h/2$. Clearly, when the can is totally full or totally empty, the total COM of the can + soda system must be at $x = 6$ cm – when the can is full, the soda is just a second cylinder whose COM is also $h/2$, and when it is empty ... we just have the can.

What about the soda? Since we know it takes a mass of soda $m_{s,f} = 1.31$ kg to fill the entire can of height h, and we know the can is a cylinder, we can find the soda's mass at any depth of filling x. Imagine the can has a cross-sectional area A. Then the density of the soda ρ must be

$$\rho = \frac{m_{s,f}}{Ah}$$

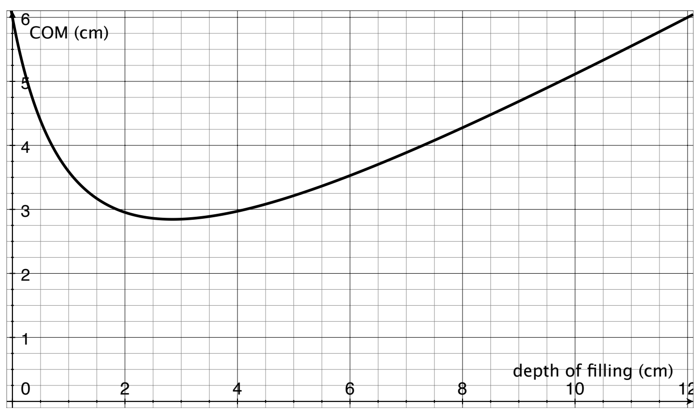
If the can is filled to a depth x, then the mass of soda in the can $m_s(x)$ is just

$$m_s(x) = \rho V = \rho(xA) = \frac{m_{s,f}x}{h} = \left(\frac{x}{h}\right) m_{s,f}$$

The result is somewhat obvious: the mass of soda in the can is the fractional filling times the total mass. If the soda is filled to a depth x, the soda itself is a cylinder of height x, and it must have a center of mass at $x/2$. Now we have the center of mass of both soda and can, we can find the center of mass of the combined system.

$$x_{com} = \frac{1}{m_c + m_s(x)}\left[\frac{h}{2}m_c + \frac{x}{2}m_s(x)\right] = \frac{m_c h + \frac{x^2}{h}m_{s,f}}{2\left[m_c + \frac{x}{h}m_{s,f}\right]} = \frac{m_c h^2 + m_{s,f}x^2}{2\left[m_c h + m_{s,f}x\right]}$$

As the plot below indicates, the COM starts at 6 cm, decreases initially as the soda begins to drain out, and rises back to 6 cm as the soda completely drains away.



3. Halliday, Resnick, & Walker, problem 9.15

At this point, we know very well how to find the peak point of the initial projectile's motion. Until the projectile splits in two, its motion is described by a parabola:

$$y(x) = x\tan\theta - \frac{gx^2}{2v_x^2}$$

Extremizing, we find the maximum height is h at horizontal distance

$$x_{\text{frag}} = \frac{v_i^2 \sin 2\theta}{2g}$$

And evaluating $y(x_{\text{frag}}) = h$, we find

$$h = \frac{v_i^2 \sin^2\theta}{2g}$$

At that point, the projectile splits into two fragments. Since no horizontal forces act, the horizontal component of the momentum is conserved. This is *not* true for the vertical component of the momentum, since the gravitational force is acting on the fragments. At the top of the trajectory, the velocity is purely horizontal, in the x direction:

$$\begin{aligned} & p_{xi} = p_i \cos\theta = m v_i \cos\theta = p_f = m_2 v_{2x} \\ \Longrightarrow \quad & v_{2x} = \frac{m}{m_2} v_i \cos\theta \end{aligned}$$

The mass m_2 is now just another projectile, and we can easily find the coordinates of its landing, given its purely horizontal velocity v_{2x} and its initial position (x_{frag}, h). Its trajectory is thus

$$y_2(x) = h - \frac{g\left(x - x_{\text{frag}}\right)^2}{2v_{2x}^2} = \frac{v_i^2 \sin^2\theta}{2g} - \frac{m_2^2 g\left(x - x_{\text{frag}}\right)^2}{2m^2 v_i^2 \cos^2\theta}$$

The second fragment will land when $y_2 = 0$:

$$\begin{aligned} y_2(x) &= \frac{v_i^2 \sin^2\theta}{2g} - \frac{m_2^2 g\left(x - x_{\text{frag}}\right)^2}{2m^2 v_i^2 \cos^2\theta} = 0 \\ x - x_{\text{frag}} &= \frac{0 \pm \sqrt{\dfrac{m_2^2 \tan^2\theta}{m^2}}}{\dfrac{m_2^2 g}{m^2 v_i^2 \cos^2\theta}} = \frac{m v_i^2 \cos^2\theta \tan\theta}{m_2 g} = \frac{m v_i^2 \cos\theta \sin\theta}{m_2 g} \end{aligned}$$

The positive root is the physical one we seek, since it corresponds to the second fragment landing further from the source than x_{frag}. Substituting $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ and our expression for x_{frag},

$$x = x_{\text{frag}} + \frac{m v_i^2 \cos\theta \sin\theta}{m_2 g} = \frac{v_i^2 \sin 2\theta}{2g} + \frac{m v_i^2 \sin 2\theta}{2m_2 g} = \left[1 + \frac{m}{m_2}\right] \frac{v_i^2 \sin 2\theta}{2g}$$

Given that fragment 2 is half the total projectile mass, or $2m_2 = m$,

$$x = \frac{3}{2} \frac{v_i^2 \sin 2\theta}{g} \approx 53\,\text{m}$$

4. Halliday, Resnick, & Walker, problem 9.33

If the elevator of mass m falls from a height $y = 36\,\text{m}$, we can find the velocity upon impact v_f easily from conservation of energy:

$$\begin{aligned} mgy &= \frac{1}{2} m v_f^2 \\ v_f &= \sqrt{2gy} \approx 27\,\text{m/s} \end{aligned}$$

(a) The impulse can be found from the total change in momentum before and after the collision, noting that the elevator starts at rest:

$$J = \Delta p = p_f - p_i = m v_f - m v_i = m\sqrt{2gy} \approx 2400\,\text{kg}\,\text{m/s}^2$$

(b) The average force is now readily found given the duration of the collision $\Delta t = 5\times 10^{-3}$ s:

$$F_{\text{avg}} = \frac{\Delta p}{\Delta t} \approx 4.8\times 10^5\,\text{N}$$

(c) If the passenger jumps upward at 7 m/s just before the collision, we need only to correct his or her final momentum in part **(a)**. Relative to the ground, the passenger now has a velocity just before the collision of $v_f - 7$ m/s, rather than v_f. After the collision ... well, the passenger's velocity is still zero one way or another.

$$J = \Delta p = p_f - p_i = mv_f - mv_i = m\left(\sqrt{2gy} - 7\,\text{m/s}\right) \approx 1800\,\text{kg}\,\text{m/s}^2$$

(d) Not much difference. Looking at the average force is only more depressing:

$$F_{\text{avg}} = \frac{\Delta p}{\Delta t} \approx 3.5\times 10^5\,\text{N}$$

Noting the passenger's weight of $W = 90g \approx 883$ N, we can estimate how many "g's" he or she pulls:

$$\begin{aligned} \text{"g's"} &= \frac{F_{\text{avg}}}{W} = \frac{a_{\text{avg}}}{g} \\ \text{no jumping:} &\approx 540\,\text{"g's"} \\ \text{with jumping:} &\approx 400\,\text{"g's"} \end{aligned}$$

Under optimistic conditions, humans can briefly survive about 100 "g's" during impact.[i] Don't bother jumping.

5. Halliday, Resnick, & Walker, problem 9.45

There are no external forces acting, so both horizontal (x) and vertical (y) momentum must be conserved. Before the explosion, we have

$$\vec{\mathbf{p}}_i = 0$$

After the explosion, we have three fragments of mass m, m and $3m$ with velocities $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_3$, respectively. We are given $\vec{\mathbf{v}}_1 = -30\,\text{m/s}\,\hat{\imath}$ and $\vec{\mathbf{v}}_2 = -30\,\text{m/s}\,\hat{\jmath}$. The third fragment has unknown speed at an unknown angle θ. We can write momentum along both axes easily, noting that mass 1 travels purely along x and mass 2 travels purely along y.

$$\begin{aligned} \vec{\mathbf{p}}_f &= m\vec{\mathbf{v}}_1 + 3m\vec{\mathbf{v}}_3 \\ p_{xf} &= mv_1 + 3mv_3\cos\theta \\ p_{yf} &= mv_2 + 3mv_3\sin\theta \end{aligned}$$

Balancing initial and final momentum,

$$\begin{aligned} 0 &= mv_1 + 3mv_3\cos\theta \\ 0 &= mv_2 + 3mv_3\sin\theta \end{aligned}$$

Rearranging, and canceling the common factor m

[i] `http://en.wikipedia.org/wiki/G-force#Human_tolerance`

$$-v_1 = 3v_3 \cos\theta = 3v_{3x}$$
$$-v_2 = 3v_3 \sin\theta = 3v_{3y}$$

Dividing these two equations, we have $\tan\theta = 1$, or $\theta = 45°$.[ii] This implies $\sin\theta = \cos\theta = 1/\sqrt{2}$. Thus,

$$v_3 = \frac{-v_1}{3\sin\theta} = \frac{-\sqrt{2}}{3} v_1 \approx 14\,\mathrm{m/s}$$

The third particle has speed $14\,\mathrm{m/s}$, moving at $45°$ relative to the x axis ($\hat{\imath}$ direction).

[ii] The other possibility mathematically is $225°$. This is perfectly valid, but will simply result in a speed of the opposite sign, meaning we will have the same velocity vector anyway.