UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 125 / LeClair Spring 2009

Problem Set 8: Solutions

1. One of the problems from the exam you did not choose.

See the separate exam II solutions,

http://faculty.mint.ua.edu/~pleclair/ph125/Exams/Ex2/ph125_s09_exam2_SOLN.pdf

2. Halliday, Resnick, & Walker, problem 10.32

(a) We are given the current period of rotation T of the pulsar in the Crab nebula as well as the rate the period changes with time, dT/dt. Using the relationship between period and angular velocity,

$$T = \frac{2\pi}{\omega} = 0.033\,\text{s}$$

Inverting, using the definition of angular acceleration α, and applying the chain rule

$$\alpha = \frac{d\omega}{dt} = 2\pi \frac{d}{dt}\left[\frac{1}{T}\right] = \frac{-2\pi}{T^2}\frac{dT}{dt} \approx -2.3\times 10^{-9}\,\text{rad/s}$$

Since $\alpha < 0$, the pulsar's rate of rotation is decreasing.

(b) The pulsar will stop when $\omega = 0$. Given the current rate of angular acceleration, and assuming it remains constant,

$$0 = \omega + \alpha t \frac{2\pi}{T} - \left(2.3\times 10^{-9}\right)t \quad \Longrightarrow \quad t \approx 8.3\times 10^{10}\,\text{s} \approx 2600\,\text{yrs}$$

The pulsar will stop after about 2600 years.

(c) The original period of rotation for the pulsar T_o can be found similarly, assuming a constant angular acceleration during its lifetime. Since its birth, $\Delta t = 2008 - 1054 \approx 955$ years have passed, or about 3×10^{10} s.

$$\frac{2\pi}{T} = \frac{2\pi}{T_o} + \alpha\Delta t$$
$$\frac{2\pi}{T_o} = \frac{2\pi}{T} - \alpha\Delta t$$
$$T_o = \frac{1}{1/T - (\alpha/2\pi)\,\Delta t} \approx 0.024\,\text{s}$$

3. Halliday, Resnick, & Walker, problem 10.42

Let total mass of all 15 discs be $M = 0.1\,\text{kg}$, and the total length of the entire assembly be $L = 1\,\text{m}$. A single disc then has mass m and radius r given by

$$m = \frac{M}{15} \qquad r = \frac{L}{2 \cdot 15}$$

Each disc has a moment of inertia about its center of mass $I_{com} = \frac{1}{2}md^2$. The central disc has only this moment of inertia, but discs away from the center of rotation must have an additional contribution to their moment of inertia based on the perpendicular distance between their centers and the center of rotation.

Looking at the whole assembly, we have 2 discs which are a distance $2r$ from the center of rotation (one on the left side, on on the right), 2 discs which are a distance $4r$ from the center of rotation, *etc.*, all the way to 2 discs a distance $14r$ from the center of rotation. The k^{th} disc from the center is a distance $2kr$ from the center of rotation ($1 \leq k \leq 7, k \in \mathbb{N}$), and its total moment of inertia is, using the parallel axis theorem,

$$I_k = \frac{1}{2}mr^2 + m\left(2kr\right)^2 = \left(\frac{1}{2} + 4k^2\right)mr^2$$

In total, we have 2 discs for each $k \in \{1, 2, 3, 4, 5, 6, 7\}$, plus the central disc. The total moment of inertia for the whole system is just the moment of inertia of the central disc, plus twice the sum of the I_k.

$$\begin{aligned}
I_{\text{tot}} &= I_{\text{central}} + 2\sum_k I_k \\
&= \frac{1}{2}mr^2 + 2\sum_{k=1}^{k=7}\left(\frac{1}{2} + 4k^2\right)mr^2 \\
&= \frac{1}{2}mr^2 + 2\frac{7}{2}mr^2 + 2\sum_{k=1}^{k=7}4k^2mr^2 \\
&= 4mr^2 + 8mr^2\sum_{k=1}^{k=7}k^2
\end{aligned}$$

Here we note that

$$\sum_{k=1}^{n}k^2 = \frac{n\left(n+1\right)\left(2n+1\right)}{6}$$

And thus

$$\begin{aligned}
I_{\text{tot}} &= 4mr^2 + 8mr^2\sum_{k=1}^{k=7}k^2 \\
&= 4mr^2 + 8\left[\frac{7\left(8\right)\left(15\right)}{6}\right]mr^2 \\
&= 4mr^2 + 8\left[\frac{840}{6}\right]mr^2 \\
&= \frac{2255}{2}mr^2 \approx 8.35 \times 10^{-6}\,\text{kg}\,\text{m}^2
\end{aligned}$$

If we have n discs in total, rather than just 15, then one can show (if n is odd)

$$I_{\text{tot}} = 4mr^2 + 8mr^2 \sum_{k=1}^{k=n} = \frac{1}{6}nmr^2 \left[2n^2+1\right]$$

If we approximate the whole system as a single rod of length L and mass M, then $I = \frac{1}{12}ML^2$. Noting that $M = 15m$ and $L = 30r$,

$$I_{\text{rod}} = \frac{1}{12}ML^2 = \frac{1}{12}(15m)(30r)^2 = \frac{2250}{2}mr^2$$

The error in this approximation is

$$\%\text{ error} = 100\left[\frac{I_{\text{tot}} - I_{\text{rod}}}{I_{\text{tot}}}\right] = 100\frac{2255-2250}{2250} \approx 0.22\%$$

4. Halliday, Resnick, & Walker, problem 10.66

We need only conservation of energy. Let the spherical shell have mass $M = 4.5\,\text{kg}$, radius $R = 0.085\,\text{m}$, moment of inertia I_1, and let it rotate at ω_1. Let the pulley have moment of inertia $I_2 = 3\times10^{-3}\,\text{kg}\,\text{m}^2$, radius $r = 0.05\,\text{m}$, with rotation at ω_2. Finally, let the hanging item have mass $m = 0.6\,\text{kg}$, traveling at velocity v after it has gone through a vertical displacement $y = 0.82\,\text{m}$ after starting from rest. Initially, we have only the gravitational potential energy of the mass. If we call the final position of interest our zero of potential energy, its initial height is then just y, and

$$E_i = mgy$$

When the mass has fallen a distance y, we have rotational energy of the pulley and sphere along with the mass' translational kinetic energy:

$$E_f = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}mv^2$$

We can relate the angular and linear velocities easily, since the string literally connects them all. If the string has not gone "slack" or broken, the linear velocity of the string must be the same at all points. The string must have the same velocity on the sphere's surface, on the pulley's surface, and at the position of the hanging mass. Thus, the string's velocity must be the same as the mass' as well:

$$v = R\omega_1 = r\omega_2$$

Substituting into our energy equation above,

$$E_f = \frac{1}{2}I_1\frac{v^2}{R^2} + \frac{1}{2}I_2\frac{v^2}{r^2} + \frac{1}{2}mv^2 = \frac{1}{2}v^2\left[\frac{I_1}{R^2} + \frac{I_2}{r^2} + m\right]$$

Conservation of energy dictates $E_f = E_i$.

$$mgy = \frac{1}{2}v^2\left[\frac{I_1}{R^2} + \frac{I_2}{r^2} + m\right]$$

$$\implies \quad v = \sqrt{\frac{2mgy}{\frac{I_1}{R^2} + \frac{I_2}{r^2} + m}}$$

Noting that for the spherical shell $I_1 = \frac{2}{3}MR^2$,

$$v = \sqrt{\frac{2mgy}{\frac{I_1}{R^2} + \frac{I_2}{r^2} + m}} = \sqrt{\frac{2mgy}{\frac{2M}{3} + \frac{I_2}{r^2} + m}} = \sqrt{\frac{2gy}{1 + \frac{2M}{3m} + \frac{I_2}{mr^2}}} \approx 1.42\,\mathrm{m/s}$$

5. Halliday, Resnick, & Walker, problem 10.65

Once again, we can use a conservation of energy approach. In the initial upright configuration, the center of mass is at a height $y_{cm,i}$, while in the final downward configuration, it is at a lower position $y_{cm,f}$. The change in the center of mass height means the system has released gravitational potential energy $m_{\text{total}}g\left(y_{cm,f} - y_{cm,i}\right) = m_{\text{tot}}g\Delta y$, where m_{total} is the sum of the hoop and rod masses, which are both simply m.

Let the $+y$ axis run vertically, with the origin at the center of rotation (i.e., at the bottom of the rod in the upright position). Let the rod have length L and the hoop radius R. By symmetry, $y_{cm,i} = -y_{cm,f}$ and $\Delta y = 2y_{cm,i}$. The initial center of mass is easily found:

$$y_{cm,i} = \frac{m\left(\frac{L}{2}\right) + m\left(R + L\right)}{2m} = \frac{1}{4}\left(3L + 2R\right)$$

Thus,

$$\Delta y = \frac{1}{2}\left(3L + 2R\right)$$

and the initial potential energy is

$$U_i = m_{\text{total}}g\Delta y = mg\left(3L + 2R\right)$$

The change in gravitational potential energy must equal the kinetic energy of the assembly at its lowest point, which is just the rotational energy of the hoop and the rod. The rod rotates about a point a distance $L/2$ from its center of mass, and thus its moment of inertia is

$$I_r = I_{com} + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2$$

The hoop rotates at a point $R + L$ from its center of mass, thus

$$I_h = I_{com} + M\left(R + L\right)^2 = \frac{1}{2}MR^2 + M\left(R + L\right)^2$$

If the system is rotating at angular velocity ω at its lowest point, which is the same for both hoop and rod, the total rotational energy of the system is

$$K_f = \frac{1}{2}I_h\omega^2 + \frac{1}{2}I_r\omega^2 = \frac{1}{2}\omega^2\left[\frac{1}{3}ML^2 + \frac{1}{2}MR^2 + M\left(R + L\right)^2\right]$$

Applying conservation of energy,

$$mg\left(3L + 2R\right) = \frac{1}{2}\omega^2\left[\frac{1}{3}ML^2 + \frac{1}{2}MR^2 + M\left(R + L\right)^2\right]$$

Noting that we are told $L = 2R$, and solving for ω

$$\omega^2 = \frac{(6L+4R)\,g}{\frac{1}{3}L^2+\frac{1}{2}R^2+(R+L)^2} = \frac{16Rg}{\frac{4}{3}R^2+\frac{1}{2}R^2+9R^2} = \frac{16g}{\frac{65}{6}R} = \frac{96g}{65R}$$
$$\omega = \sqrt{\frac{96g}{65R}}$$

Given $R=0.15\,\text{m}$, $\omega \approx 9.8\,\text{rad/s}$.

6. Halliday, Resnick, & Walker, problem 10.91

A solid cylinder about its central axis has $I_c = \frac{1}{2}MR_c^2$, while a hoop has $I_h = MR_h^2$. Clearly, we can only have $I_c = I_h$ if $R_h = R_c/\sqrt{2}$.

One can show that an equivalent hoop can always be found simply by dimensional analysis. For any body, the moment of inertia is defined as

$$I = \int r^2\,dm$$

This means I must have units of [kg][m]. If the body of interest has a mass M, we can always find a characteristic distance k such that $I = Mk^2$. We could make this slightly more rigorous, but there is really not much point: the main point is simple and qualitative, the "proof" should not imply more.

Really, this is a proof of nothing, when you think about it ... the end result is that all moments of inertia have the same units, so they will always look like some mass times some distance squared. If we use the real mass, we can find an "effective distance," and similarly if we use a real characteristic distance, we could find an "effective mass." We prefer the former, since realistic objects have easily characterized masses, while their shapes are usually terribly unfortunate.

Requiring a hoop of the same mass to have the same moment of inertia,

$$k^2M = MR^2 = I \quad\Longrightarrow\quad k = \sqrt{I}M$$

In the case of a solid cylinder about its center of mass, the characteristic distance would be $k=1/\sqrt{2}$.

7. Halliday, Resnick, & Walker, problem 10.7

There are 8 divisions to the wheel, which means each covers $\pi/4$ radians. The angular velocity of the wheel is

$$\omega = \frac{2.5\,\text{rev}}{1\,\text{s}}\left(\frac{2\pi\,\text{rad}}{1\,\text{rev}}\right) = 5\pi\,\text{rad}s$$

If the arrow has length d and velocity v, it takes $t=d/v$ seconds to pass through the wheel completely. In that time t, the wheel must not rotate more than $\pi/4$ radians if the arrow is to pass cleanly through. Thus,

$$\omega < \frac{\pi/4}{t} = \frac{\pi v}{4d} \quad\Longrightarrow\quad v > \frac{4wd}{\pi} \approx 4\,\text{m/s}$$

Watch the units ... it can be tricky when radians are involved, since they are technically dimensionless.