UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 125 / LeClair Spring 2009

Problem Set 9 Solutions

1. A particle begins at the origin and moves successively in the following directions:

- 1 unit to the right $(+x)$
- $\frac{1}{2}$ unit up $(+y)$
- $\frac{1}{4}$ unit to the right
- $\frac{1}{8}$ unit down
- $\frac{1}{16}$ unit to the right
- *etc.*

The length of each move is half the length of the previous move, and movement continues in the "zigzag" manner described. Find the coordinates of the point to which the zigzag converges.

Solution: Consider the x and y coordinates separately. The increasing x and y coordinates can each be described by a geometric series.

$$x = 1 + \frac{1}{4} + \frac{1}{16} + \ldots = \left(\frac{1}{4}\right)^0 + \left(\frac{1}{4}\right)^1 + \left(\frac{1}{4}\right)^2 + \ldots = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$$

$$y = \frac{1}{2} + \frac{-1}{8} + \ldots = \frac{1}{2}\left[1 - \frac{1}{4} + \ldots\right] = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n$$

Both series have the general form

$$\sum_{n=0}^{\infty} az^n \qquad (|z| < 1)$$

Now we must prove that series of this sort have a finite sum, and evaluate that sum. We will sketch a relatively general proof. Let a and z be complex numbers $a, z \in \mathbb{C}$. First consider the finite geometric series

$$x(n) \equiv az^n \qquad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$$

where n is a non-negative integer. The partial sum of this series through the first N terms can be defined by[i]

$$S_N(z) = \sum_{n=0}^{N-1} az^n = a\left(1 + z + z^2 + z^3 + \ldots + z^{N+1}\right)$$

The partial sum $S_N(z)$ can be evaluated by finding the difference between $S_N(z)$ and $zS_N(z)$:

[i]The limit of the summation is $N-1$ to give the first N terms since we start at $n=0$.

$$
\begin{aligned}
S_N(z) &\equiv a\left(1+z+z^2+\ldots+z^{N-1}\right)\\
zS_N(z) &= a\left(\quad\; z+z^2+\ldots+z^N\right)\\
\Longrightarrow \quad S_N(z)-zS_N(z) &= a\left(1-z^N\right)\\
\Longrightarrow \quad S_N(z) &= \frac{a\left(1-z^N\right)}{1-z} \qquad (z\neq 1)
\end{aligned}
$$

If $z=1$, the partial sum is $S_N(1)=N$ by inspection. For an *infinite* geometric series, we must only evaluate the limit of the partial sum as N approaches infinity.

$$
\lim_{N\to\infty} S_N(z) = \lim_{N\to\infty}\frac{a\left(1-z^N\right)}{1-z} = \lim_{N\to\infty}\left(\frac{a}{1-z}-\frac{az^N}{1-z}\right) = \left(\frac{a}{1-z}\right)-\left(\frac{a}{1-z}\right)\lim_{N\to\infty} z^N
$$

$$
\lim_{N\to\infty} S_N(z) = \begin{cases} \dfrac{a}{1-z} & (|z|<1)\\ \pm\infty & (|z|\geq 1)\end{cases}
$$

We see that the infinite geometric series is (absolutely) convergent provided $|z|<1$, since

$$
\lim_{N\to\infty} z^N = 0 \qquad (|z|<1)
$$

Applied to the present case, we see for the x coordinate that $a_x=1$ and $z_x=\frac{1}{4}$ and for the y coordinate $a_y=\frac{1}{2}$ and $z_y=-\frac{1}{4}$. Thus,

$$
x = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}
$$

$$
y = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-1}{4}\right)^n = \frac{\frac{1}{2}}{1-\frac{-1}{4}} = \frac{2}{5}
$$

Our particle therefore converges to the coordinates $\left(\frac{4}{3},\frac{2}{5}\right)$.

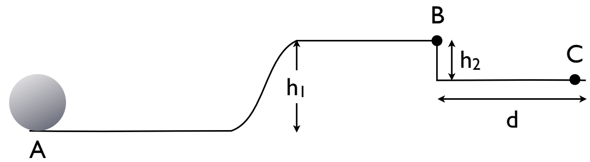
N.B. – This was a 1984 American Regions Mathematics League (ARML) competition question.

2. Halliday, Resnick, & Walker, problem 11.14

Solution: Our sphere starts out at point A in the sketch below already undergoing smooth rolling motion, with center of mass velocity v_i. Since the sphere rolls without slipping, its angular and linear velocities must be related by the sphere's radius R, $v_i=R\omega$. We can apply conservation of mechanical energy to find the sphere's velocity at point B. Let the zero of gravitational potential energy be the lowest level in the diagram (the height of point A). At A, the total mechanical energy is purely kinetic, with both linear and rotational terms:

$$
K_A+U_A = \frac{1}{2}mv_i+\frac{1}{2}I\omega_i^2 = \frac{1}{2}mv_i+\frac{1}{2}I\frac{v_i^2}{R} = \frac{1}{2}v_i^2\left(m+\frac{I}{R^2}\right)
$$

At point B, we also have translational and rotational kinetic energy, characterized by linear and angular velocities



v_b and ω_b, respectively. We still have $v_b = R\omega_b$, since the motion is purely rolling without slipping. We also have now a gravitational potential energy mgh_1, and

$$K_B + U_B = \frac{1}{2}v_b^2\left(m + \frac{I}{R^2}\right) + mgh_1$$

Applying conservation of energy between A and B, we can solve for v_i:

$$\begin{aligned} K_A + U_A &= K_B + U_B \\ \frac{1}{2}v_i^2\left(m + \frac{I}{R^2}\right) &= \frac{1}{2}v_b^2\left(m + \frac{I}{R^2}\right) + mgh_1 \\ v_i^2 &= v_b^2 + \frac{2mgh_1}{m + I/R^2} \end{aligned}$$

We need only an expression for v_b. At point B, the sphere is launched from height h_2 above the far right platform, and it behaves just as any other projectile. In the absence of air resistance, the rate of rotation ω will not change from B to C, and we can therefore ignore the rotational motion. The sphere covers a horizontal distance d in a time t after being launched horizontally at v_b, and it covers a vertical distance h_2 in the same time t under the influence of gravity. Thus,

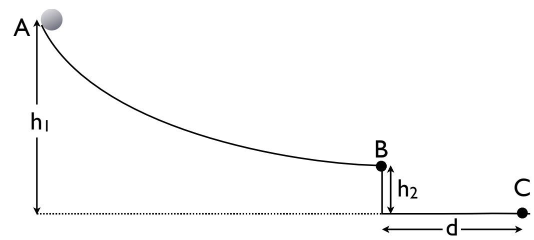
$$\begin{aligned} d &= v_b t \\ -h_2 &= -\frac{1}{2}gt^2 \\ \Longrightarrow \qquad v_b &= d\sqrt{\frac{g}{2h_2}} \end{aligned}$$

Using this result in our expression above, and noting $I = \frac{2}{5}mr^2$ for a solid sphere,

$$\begin{aligned} v_i^2 &= v_b^2 + \frac{2mgh_1}{m + I/R^2} = \frac{d^2g}{2h_2} + \frac{2mgh_1}{m + I/R^2} \\ v_i^2 &= \frac{d^2g}{2h_2} + \frac{2mgh_1}{m + \frac{2}{5}m} = \frac{d^2g}{2h_2} + \frac{2gh_1}{\frac{7}{5}} = \frac{d^2g}{2h_2} + \frac{10}{7}gh_1 \\ v_i &= \sqrt{\frac{d^2g}{2h_2} + \frac{10}{7}gh_1} \approx 1.34\,\text{m/s} \end{aligned}$$

3. Halliday, Resnick, & Walker, problem 11.16

Solution: First, a simple sketch for reference:



Once again, we need only apply conservation of energy. The object starts out at A with only gravitational potential energy, and at B has gained rotational and translational kinetic energy. Since we have rolling motion without slipping, we can relate linear and angular velocities at B via $v=R\omega$. Let the zero for gravitational potential energy by the lowest level in the figure (that of C). Conservation of energy between A and B yields:

$$\begin{aligned}
mgH &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgh \\
mg\left(H-h\right) &= \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{R^2} = \frac{1}{2}v^2\left(m+\frac{I}{R^2}\right) = \frac{1}{2}v^2\left(m+\beta m\right) = \frac{1}{2}mv^2\left(1+\beta\right) \\
1+\beta &= \frac{2g\left(H-h\right)}{v^2} \\
\beta &= \frac{2g\left(H-h\right)}{v^2} - 1
\end{aligned}$$

We need only an expression for v. Just as in the previous problem, we can use the equations of projectile motion.

$$\begin{aligned}
d &= vt \\
-h &= -\frac{1}{2}gt^2 \\
\Longrightarrow \qquad v &= d\sqrt{\frac{g}{2h}}
\end{aligned}$$

Thus,

$$\beta = \frac{2g\left(H-h\right)}{v^2} - 1 = \frac{2g\left(H-h\right)}{\frac{d^2 g}{2h}} - 1 = \frac{4h\left(H-h\right)}{d^2} - 1 \approx 0.25$$

4. Halliday, Resnick, & Walker, problem 11.35

Solution: This problem is sneakier than it seems on first sight, since we don't know the net force involved. Without that, we can't simply use the definition of torque $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$. However, we *can* calculate the angular momentum $\vec{\mathbf{L}}$ with the quantities given, and its time derivative gives us the torque.

$$\begin{aligned}
\vec{\mathbf{L}} &= \vec{\mathbf{r}} \times \vec{\mathbf{p}} = m\left(\vec{\mathbf{r}} \times \vec{\mathbf{v}}\right) = m\left(\vec{\mathbf{r}} \times \frac{d\vec{\mathbf{r}}}{dt}\right) \\
\vec{\mathbf{v}} &= \frac{d\vec{\mathbf{r}}}{dt} = 8.0t\,\hat{\boldsymbol{\imath}} - \left(2.0+12t\right)\,\hat{\boldsymbol{\jmath}}
\end{aligned}$$

It is marginally easier to first calculate the cross product:

$$\vec{\mathbf{r}}\times\vec{\mathbf{v}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ r_x & r_y & r_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 4.0t^2 & -\left(2.0t+6.0t^2\right) & 0 \\ 8.0t & -\left(2.0+12t\right) & 0 \end{vmatrix} = \left[-4.0t^2\left(2.0+12t\right)+8.0t\left(2.0t+6.0t^2\right)\right]\hat{\mathbf{k}} = 8t^2\,\hat{\mathbf{k}}$$

Thus,

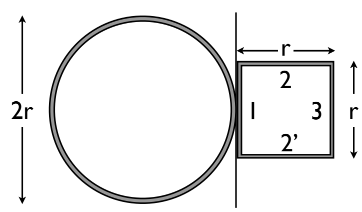
$$\vec{\mathbf{L}} = 8mt^2\,\hat{\mathbf{k}}$$
$$\vec{\boldsymbol{\tau}} = \frac{d\vec{\mathbf{L}}}{dt} = 16mt\,\hat{\mathbf{k}} = 48t\,\hat{\mathbf{k}}$$

It is comforting that $\vec{\mathbf{L}}$ and $\vec{\boldsymbol{\tau}}$ are both perpendicular to the plane formed by $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$.

By inspection, we can see that angular momentum is *increasing* for all $t \geq 0$.

5. Halliday, Resnick, & Walker, problem 11.41

Solution: Again, a quick sketch:



The square is made up of four thin rods of length r, while the hoop has radius r. First, we calculate the moment of inertia of the square. The first rod labeled "1" is on the axis of rotation. If its thickness is negligible, its moment of inertia is essentially zero – all the mass is at distance zero from the axis of rotation. The horizontal rods 2 and $2'$ are both rotating about a distance $r/2$ from their center of mass, and thus

$$I_2 = I_{2'} = I_{com} + m\left(\frac{r}{2}\right)^2 = \frac{1}{12}mr^2 + \frac{1}{4}mr^2 = \frac{1}{3}mr^2$$

The rod labeled 3 has all its mass located a distance r from the axis of rotation (still presuming the thickness to be negligible), and thus its moment of inertia is the same as that of a particle of mass m a distance r from the axis of rotation, $I_3 = mr^2$. In total,

$$I_{\square} = I_1 + I_2 + I_{2'} + I_3 = 0 + \frac{1}{3}mr^2 + \frac{1}{3}mr^2 + mr^2 = \frac{5}{3}mr^2$$

Our hoop rotates a distance r from its center of mass, and thus

$$I_{\circ} = I_{com} + mr^2 = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2$$

The total system then has

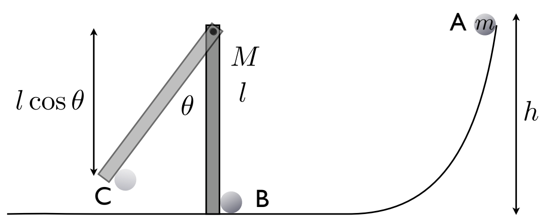
$$I_{\text{tot}} = I_{\square} + I_{\circ} = \left(\frac{5}{3} + \frac{3}{2}\right) mr^2 = \frac{19}{6} mr^2 \approx 1.6\,\text{kg}\,\text{m}^2$$

The total angular momentum can be found from the moment of inertia and the angular velocity, the latter of which can be found easily from the period of rotation:

$$\omega = \frac{2\pi}{T}$$
$$L = I_{\text{tot}}\omega = \frac{2\pi I_{\text{tot}}}{T} = \frac{19\pi mr^2}{3T} \approx 4.0\,\text{kg}\,\text{m}^2/\text{s}$$

6. Halliday, Resnick, & Walker, problem 11.66

Solution: Again, a quick sketch. Let A be the starting point, B the moment of collision between the ball and rod, and C the point when maximum height is reached by the rod + ball system. We approximate the ball as a point mass, since we are told it is small (and we anyway have no way of calculating its moment of inertia, since we do not have any geometrical details ...).



The velocity v of the ball at point B can be found using conservation of mechanical energy. Let the floor be the height of zero gravitational potential energy.

$$K_A + U_A = K_B + U_B$$
$$mgh = \frac{1}{2}mv^2$$
$$\Longrightarrow v = \sqrt{2gh}$$

The collision is clearly inelastic, since the ball sticks to the rod. We could use conservation of linear momentum, but this would require breaking up the rod into infinitesimal discrete bits of mass and integrating over its length. Easier is to use conservation of *angular* momentum about the pivot point of the rod. Just before the collision, we have the ball moving at speed v a distance l. Let $\hat{\boldsymbol{\imath}}$ be to the right, and $\hat{\boldsymbol{\jmath}}$ upward (making $\hat{\mathbf{k}}$ into the page). The initial angular momentum is then

$$\vec{\mathbf{L}}_i = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = l\,\hat{\boldsymbol{\jmath}} \times (-mv\,\hat{\boldsymbol{\imath}}) = -mvl\,(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}}) = mvl\,\hat{\mathbf{k}} = ml\sqrt{2gh}\,\hat{\mathbf{k}}$$

After the collision, we have the rod and mass stuck together, rotating at angular velocity ω. Defining counterclockwise rotation to be positive as usual, the final angular momentum is thus

$$\vec{\mathbf{L}}_f = I\omega\,\hat{\mathbf{k}}$$

The total moment of inertia about the pivot point is that of the rod rotating plus that of the ball. The rod rotates a distance $l/2$ from its center of mass, and again we approximate the ball as a point mass rotating at a distance l (since we told it is small).

$$I = I_{\text{rod}} + I_{\text{ball}} = I_{\text{rod, com}} + M\left(\frac{l}{2}\right)^2 + ml^2 = \frac{1}{12}Ml^2 + Ml^2 + ml^2 = \left(\frac{1}{3}M + m\right)l^2$$

Equating initial and final angular momentum, we can solve for the angular velocity after the collision:

$$L_f = I\omega = L_i = mvl = ml\sqrt{2gh}$$
$$\left(\frac{1}{3}M + m\right)l^2\omega = ml\sqrt{2gh}$$
$$\omega = \frac{m\sqrt{2gh}}{\left(\frac{1}{3}M + m\right)l}$$

At this point, we may use conservation of energy once again. When the system reaches its maximum angle θ at C, the center of mass of the rod + ball system will have moved up by an amount Δy_{cm}. The change in gravitational potential energy related to this change in center of mass height must be equal to the rotational kinetic energy just after the collision. Thus,

$$\frac{1}{2}I\omega^2 = \frac{\vec{\mathbf{L}}\cdot\vec{\mathbf{L}}}{2I} = \frac{L^2}{2I} = (m + M)\,g\Delta y_{cm}$$

Here we have noted that the rotational kinetic energy can be related to the angular momentum to save a bit of algebra. To proceed, we must find the difference in the center of mass height between points C and B. Let $y=0$ be the height of the floor. At point B,

$$y_{cm,B} = \frac{M\left(\frac{L}{2}\right) + m\,(0)}{m + M} = \left(\frac{l}{2}\right)\left(\frac{M}{m + M}\right)$$

At point C, the ball is now at a height $l - l\cos\theta$, while the center of mass of the rod (its midpoint) is now at $l - l\cos\theta + \frac{1}{2}l\cos\theta$. Thus,

$$y_{cm,C} = \frac{M\left(l - l\cos\theta + \frac{1}{2}l\cos\theta\right) + m\,(l - l\cos\theta)}{m + M} = \frac{Ml\left(1 - \frac{1}{2}\cos\theta\right) + ml\,(1 - \cos\theta)}{m + M}$$

The change in center of mass height can now be found:

$$\begin{aligned}\Delta y_{cm} = y_{cm,C} - y_{cm,B} &= \frac{Ml\left(1 - \frac{1}{2}\cos\theta\right) + ml\,(1 - \cos\theta) - \frac{1}{2}Ml}{m + M} \\ &= \frac{\frac{1}{2}Ml\,(1 - \cos\theta) + ml\,(1 - \cos\theta)}{m + M} \\ &= \frac{l}{m + M}\,(1 - \cos\theta)\left(m + \frac{1}{2}M\right)\end{aligned}$$

Using our previous energy balance between B and C,

$$\frac{L^2}{2I} = (m+M)\,g\Delta y_{cm} = lg\,(1-\cos\theta)\left(m+\frac{1}{2}M\right)$$

Since the initial and final angular momenta are equal, we may substitute either L_f or L_i, the latter being the easiest option. This is not strictly *necessary* – we could use L_f or even just grind through $\frac{1}{2}I\omega^2$ and the result must be the same. However, using L_i here saves quite a bit of algebra in the end when we try to put θ in terms of only given quantities. Doing so, and solving for θ

$$\frac{L_f^2}{2I} = \frac{L_i^2}{2I} = \frac{2l^2m^2gh}{2\left(\frac{1}{3}M+m\right)l^2} = lg\,(1-\cos\theta)\left(m+\frac{1}{2}M\right)$$

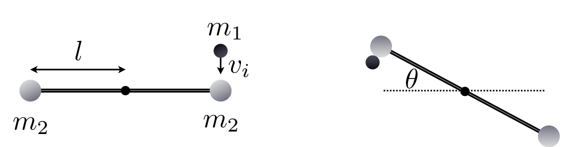
$$1-\cos\theta = \frac{m^2h}{l\left(\frac{1}{3}M+m\right)\left(\frac{1}{2}M+m\right)}$$

$$\theta = \cos^{-1}\left[1-\frac{m^2h}{l\left(\frac{1}{3}M+m\right)\left(\frac{1}{2}M+m\right)}\right] \approx 32^\circ$$

Note that for $m=0$, $\theta=0$, as we expect. On the other hand, for $M=0$ we have $\cos\theta=1-h/l=1/2$. This means that the particle is at a height $l-l\cos\theta=l/2=h$ at point C – exactly what we would expect if mechanical energy were conserved!

7. Halliday, Resnick, & Walker, problem 11.67

Solution: Again, a quick sketch.



(a) Our dumbbell, consisting of two masses m_2 both a distance l from its center of mass, is struck by a smaller mass m_1 traveling at velocity $\vec{\mathbf{v}}_i$. Conservation of angular momentum can be used to find the angular velocity after the collision. Before the collision, with $\hat{\imath}$ to the right and $\hat{\jmath}$ upward, we have the smaller mass' momentum $\vec{\mathbf{p}}_i=-m_1v_i\,\hat{\jmath}$ acting at a distance $\vec{\mathbf{r}}=l\,\hat{\imath}$ from the center of rotation.

$$\vec{\mathbf{L}}_i = \vec{\mathbf{r}}\times\vec{\mathbf{p}} = -m_1lv_i\,\hat{\mathbf{k}}$$

The minus sign indicates a clockwise rotation following our usual convention, which is sensible. After the collision, the entire system rotates clockwise at angular velocity $\vec{\boldsymbol{\omega}}=-\omega\,\hat{\mathbf{k}}$. The total moment of inertia is found easily, since we have only point-like masses:

$$I = \sum_i m_ir_i^2 = m_2l^2+m_2l^2+m_1l^2 = l^2\,(2m_2+m_1)$$

The final angular momentum is then

$$\vec{\mathbf{L}}_f = I\vec{\boldsymbol{\omega}} = -l^2\omega\left(2m_2 + m_1\right)\hat{\mathbf{k}}$$

Conservation of angular momentum gives us

$$\vec{\boldsymbol{\omega}} = \frac{L_i}{I} = \frac{m_1 v_i}{\left(2m_2 + m_1\right)l}\,\hat{\mathbf{k}} \approx 0.15\,\text{rad/s}\,\hat{\mathbf{k}}$$

(b) The initial kinetic energy of the system is only that of the smaller mass, $K_i = \frac{1}{2}m_1 v_i^2$. The final kinetic energy is the rotational kinetic energy of the whole system, which is simplified a bit in terms of angular momentum

$$K_f = \frac{1}{2}I\omega^2 = \frac{\vec{\mathbf{L}}\cdot\vec{\mathbf{L}}}{2I} = \frac{L_i^2}{2I} = \frac{m_1^2 l^2 v_i^2}{2l^2\left(2m_2 + m_1\right)} = \frac{1}{2}m_1 v_i^2\left(\frac{m_1}{2m_2 + m_1}\right) = K_i\left(\frac{m_1}{2m_2 + m_1}\right)$$

Note that since angular momentum is conserved, we can use either L_i or L_f in the kinetic energy equation; using L_i is somewhat simpler algebraically. The ratio of final to initial kinetic energies is thus

$$\frac{K_f}{K_i} = \frac{m_1}{2m_2 + m_1} \approx 0.0123$$

(c) What happens once the system starts rotating? Even without the initial kinetic energy of the smaller mass, since all forces present after the collision are conservative the whole system would have enough energy to rotate through 180°, since that would put all of the masses back at the same height. The gravitational potential energy of the system right after the collision is the same as that after rotating through 180°, so the system must rotate at least that much.

After rotating through 180°, the total mechanical energy of the system is unchanged from the point right after the collision. The system will continue rotating through a further maximum angle θ at which point the gain in potential energy equals the kinetic energy right after the collision, K_f. As the system rotates, one of the m_2 masses will go up by an amount $h = l\sin\theta$, and the other m_2 mass will go down by the same amount. The only change in potential energy comes from the smaller m_1 mass moving up by h! We can balance mechanical energy between configurations right after the collision, after rotating through 180°, and after rotating through an additional angle θ. Let the initial horizontal axis of the dumbbell be the zero of potential energy.

$$\begin{aligned}
\text{after collision:} &\quad K + U = K_f \\
\text{after rotating through } 180^\circ\text{:} &\quad K + U = K_f \\
\text{after an additional rotation by } \theta\text{:} &\quad K + U = m_2 gl\sin\theta + m_1 gl\sin\theta - m_2 gl\sin\theta = m_1 gl\sin\theta \\
\text{conservation of mechanical energy} \quad \Longrightarrow &\quad m_1 gl\sin\theta = K_f = \frac{m_1^2 v_i^2}{2\left(2m_2 + m_1\right)} \\
&\quad \sin\theta = \frac{m_1 v_i^2}{2gl\left(2m_2 + m_1\right)} \\
&\quad \theta = \sin^{-1}\left[\frac{m_1 v_i^2}{2gl\left(2m_2 + m_1\right)}\right] \approx 1.3^\circ
\end{aligned}$$

The total angle of rotation is thus $180^\circ + 1.3^\circ = 181.3^\circ$.