Problem Set 1 Solutions

Daily problems due 13 Jan 2014

1. Here are two vectors:

$$\vec{\mathbf{a}} = 1.0\,\hat{\imath} + 2.0\,\hat{\jmath} \qquad \vec{\mathbf{b}} = 3.0\,\hat{\imath} + 4.0\,\hat{\jmath}$$

Find the following quantities:

a) the magnitude of $\vec{\mathbf{a}}$
b) the angle of $\vec{\mathbf{a}}$ relative to $\vec{\mathbf{b}}$
c) the magnitude and angle of $\vec{\mathbf{a}}+\vec{\mathbf{b}}$
d) the magnitude and angle of $\vec{\mathbf{a}}-\vec{\mathbf{b}}$

Solution: Since this is a purely mathematical problem, we'll forgo the problem solving template.

(a) If $\vec{\mathbf{a}}$ is in general a vector defined by $\vec{\mathbf{a}}=a_x\hat{\imath}+a_y\hat{\jmath}$, the magnitude of $\vec{\mathbf{a}}$ is found by

$$|\vec{\mathbf{a}}| = \sqrt{a_x^2 + a_y^2} \tag{1}$$

In the present case, this gives $|a|=\sqrt{5}$.

(b) The angle θ between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is most easily found using the scalar product:

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta = a_xb_x + a_yb_y \tag{2}$$

Thus,

$$\cos\theta = \frac{a_xb_x + a_yb_y}{|\vec{\mathbf{a}}||\vec{\mathbf{b}}|} = \frac{3+8}{\left(\sqrt{5}\right)(5)} = \frac{11}{5\sqrt{5}} \tag{3}$$

$$\theta = \cos^{-1}\left(\frac{11}{5\sqrt{5}}\right) \approx 10.3^\circ \tag{4}$$

(c) First we'll need the vector sum

$$\vec{\mathbf{a}}+\vec{\mathbf{b}} = (a_x+b_x)\,\hat{\imath} + (a_y+b_y)\,\hat{\jmath} = 4\hat{\imath} + 6\hat{\jmath} \tag{5}$$

The magnitude is easy enough

$$|\vec{\mathbf{a}}+\vec{\mathbf{b}}| = \sqrt{(a_x+b_x)^2+(a_y+b_y)^2} = \sqrt{4^2+6^2} = \sqrt{52} = 2\sqrt{13} \tag{6}$$

The tangent of the angle with the horizontal axis is the ratio of the y and x components:

$$\tan\theta = \frac{a_y+b_y}{a_x+b_x} = \frac{6}{3} = 2 \tag{7}$$

$$\theta = \tan^{-1} 2 = 63.4° \tag{8}$$

(d) Same deal, but we reverse the sign of $\vec{\mathbf{b}}$ to perform subtraction.

$$\vec{\mathbf{a}}-\vec{\mathbf{b}} = \vec{\mathbf{a}}+(-\vec{\mathbf{b}}) = (a_x-b_x)\,\hat{\boldsymbol{\imath}}+(a_y-b_y)\,\hat{\boldsymbol{\jmath}} = -2\hat{\boldsymbol{\imath}}+-2\hat{\boldsymbol{\jmath}} \tag{9}$$

The magnitude proceeds as it did in the last part ...

$$|\vec{\mathbf{a}}-\vec{\mathbf{b}}| = \sqrt{(a_x-b_x)^2+(a_y-b_y)^2} = \sqrt{(-2)^2+(-2)^2} = \sqrt{8} = 2\sqrt{2} \tag{10}$$

...as does the determination of the angle:

$$\tan\theta = \frac{a_y-b_y}{a_x-b_x} = \frac{-2}{-2} = 1 \tag{11}$$

$$\theta = \tan^{-1} 1 = 45° \tag{12}$$

The angle the vector makes with respect to the horizontal axis is 45°. We should be careful, however: knowing that both components are negative, we know the vector points down and to the left. This tells us that the vector is pointing 45° *below* the horizontal axis, and backward along the $-y$ direction. If we want to be more precise, we might say the vector makes an angle of 225° with the x axis.

2. (a) If the position of a particle is given by $x=20t-5t^3$, where x is in meters and t is in seconds, when, if ever, is the particle's velocity zero? (b) When is its acceleration a zero? (c) For what time range (positive or negative) is a negative? (d) Positive? (e) Sketch graphs of $x(t)$, $v(t)$, and $a(t)$.

Solution: Let's try to apply the problem template in solving this one.

Find/given: We are given a particle's position as a function of time $x(t)$. We are asked to find a number of things: when the particle's velocity v is zero, when its acceleration a is zero, and for what times the acceleration is positive or negative. Additionally, we should sketch the position, velocity, and acceleration as a function of time.

Sketch: All we can really do in this case starting out is plot the function $x(t)$. See the end of the solutions for details on how I made the plot, if you are curious.

We know $v(t)$ is the slope of the $x(t)$ curve for a given time t, so we can already see that the velocity

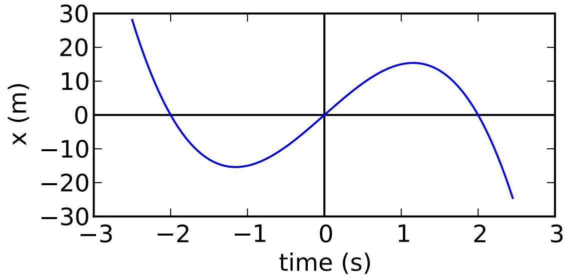


Figure 1: *Plot using simple Python code*

is zero at two different times.

Relevant equations: We will need to know the relationships between position, velocity, and acceleration:

$$v = \frac{dx}{dt} \tag{13}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{14}$$

Symbolic solution: We require this for parts a-e. **(a)** We can find the velocity by noting $v=dx/dt$:

$$v(t) = \frac{dx}{dt} = 20 - 15t^2 = 0 \tag{15}$$

$$15t^2 = 20 \tag{16}$$

$$t^2 = \frac{20}{15} = \frac{4}{3} \tag{17}$$

$$t = \pm\frac{2}{\sqrt{3}} \tag{18}$$

There is nothing from the setup of the problem to suggest that the given position is not valid for times less than zero, so we must conclude that there are two such times: $\pm 2/\sqrt{3}\,\mathrm{s}$.

(b) The acceleration is found from $a=dv/dt$:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -30t \tag{19}$$

Clearly, $a=0$ only at $t=0$.

(c,d) Given $a=-30t$, a is negative for all $t<0$ and positive for all $t>0$.

(e) Knowing how to relate position, velocity, and acceleration, we can plot them all together:

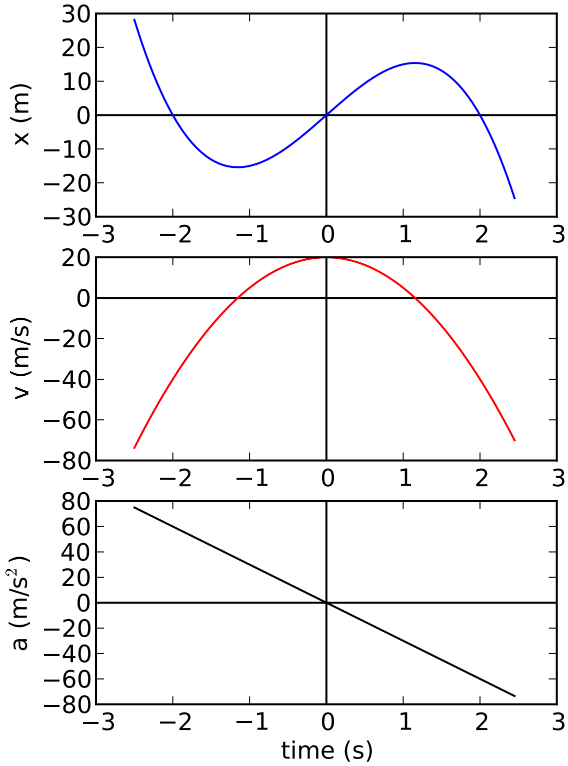


Figure 2: *Plot using simple Python code*

Numerical solution: There is not much to do in this case, since the problem inherently involved numbers in the first place, except to note that for part (a) $t = 2/\sqrt{3} \approx \pm 1.2\,\mathrm{s}$.

Double Check: In this case, the problem is purely mathematical: the fact that the math works out is really enough. One simple double check is to verify that the plot agrees with the mathematics. We see that velocity is zero where the position is at a minimum or maximum, and this seems to happen at about 1.2 s on the graph. We also note that the acceleration is positive where the velocity curve has positive slope and negative where the velocity curve has negative slope.

Daily problem due 15 Jan 2014:

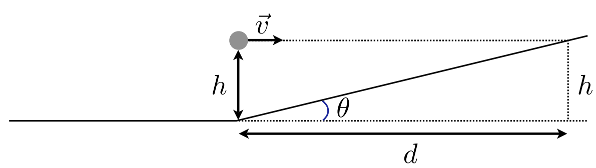
3. A pilot flies horizontally at 1300 km/h, at height $h = 35\,\mathrm{m}$ above initially level ground. However, at time $t = 0$, the pilot begins to fly over ground sloping upward at angle $\theta = 4.3°$. If the pilot does not change the airplane's heading, at what time t does the plane strike the ground?

Solution: This is problem 2.80 from your textbook (at least in the edition I have).

Given: The initial velocity and height of a plane flying toward an upward slope of angle θ.

Find: How long before the plane hits the slope? At time $t = 0$, the plane is at the beginning of the slope, a height h above level ground. Assuming the plane continues at the same horizontal speed, we wish to find the time at which the plane hits the slope. Given the plane's velocity and height and the slope's angle, we can relate the horizontal distance to intercept the ramp to the plane's height.

Sketch: Assume a spherical plane (it doesn't matter). If the plane is at altitude h, it will hit the ramp after covering a horizontal distance d, where $\tan\theta = h/d$.



Relevant equations: We can relate the horizontal distance to intersect the ramp to the plane's altitude using the known slope of ground:

$$\tan\theta = \frac{h}{d}$$

We can determine how long the horizontal distance d will be covered given the plane's constant horizontal speed v:

$$d = vt$$

Symbolic solution: Combining our equations above, the time t it takes for the plane to hit the slope is

$$t = \frac{d}{v} = \frac{h}{v\tan\theta}$$

Numeric solution: Using the numbers given, and converting units,

$$t = \frac{h}{v\tan\theta} = \frac{35\,\mathrm{m}}{1300\,\mathrm{km/h}\,(1000\,\mathrm{m/km})\,(1\,\mathrm{h}/3600\,\mathrm{s})\,(\tan 4.3°)} \approx 1.3\,\mathrm{s}$$

The problems below are due by the end of the day on 17 Jan 2014.

4. (a) With what speed must a ball be thrown vertically from the ground level to rise to a maximum height of 50 m? **(b)** How long will it be in the air? **(c)** Sketch graphs of y, v, and a versus t for the ball. On the first two graphs, indicate the time at which 50 m is reached.

Solution: **Given:** The maximum height the ball will reach, $y_{\text{max}} = 50\,\text{m}$.

Find: The initial speed v_{iy} required for a ball thrown vertically upward to reach a height of 50 m and the total time the ball remains in the air.

Sketch: A ball is thrown straight up in the air, you can probably use your imagination here. Let the y axis run vertically, with the $+y$ direction being upward and the starting position of the ball at $y=0$. That makes the acceleration due to gravity $-g$, as it points downward. Let time $t=0$ be the moment the ball is released.

Relevant equations: We will need only our equation for position under a constant acceleration of $-g$ and a definition of velocity. Since the ball starts at $y=0$, our position equation is somewhat simpler.

$$y(t) = v_{iy}t - \frac{1}{2}gt^2 \tag{20}$$

$$v_y(t) = \frac{dy}{dt} = v_{iy} - gt \tag{21}$$

Symbolic solution: At maximum height, the velocity is instantaneously zero. This bit of knowledge lets us find out at what time t_{max} the ball reaches maximum height, and then we can use the position equation to get the initial velocity.

$$v_y(t_{\text{max}}) = v_{iy} - gt_{\text{max}} = 0 \qquad \Longrightarrow \qquad t_{\text{max}} = \frac{v_{iy}}{g} \tag{22}$$

Already this solves the second portion of the problem: this is how long the ball takes to reach maximum height, it takes the same amount of time to come back down. Thus, the ball spends a net amount of time $\Delta t = 2t_{\text{max}}$ in the air.

The position at time t_{max} is known to be $y_{\text{max}} = 50\,\text{m}$. If we know the time and the position, the only unknown remaining in the position equation is the initial velocity.

$$y(t_{\text{max}}) = y_{\text{max}} = v_{iy}\left(\frac{v_{iy}}{g}\right) - \frac{1}{2}g\left(\frac{v_{iy}}{g}\right)^2 = \frac{v_{iy}^2}{2g} \tag{23}$$

$$v_{iy}^2 = 2gy_{\text{max}} \tag{24}$$

$$v_{iy} = \sqrt{2gy_{\text{max}}} \tag{25}$$

Numerical Solution: Plugging in the numbers given,

$$v_{iy} = \sqrt{2gy_{\text{max}}} \approx \sqrt{2\,(9.8\,\text{m/s}^2)\,(50\,\text{m})} \approx 31\,\text{m/s} \tag{26}$$

$$\Delta t = \frac{2v_{iy}}{g} \approx \frac{2\,(31\,\text{m/s})}{9.8\,\text{m/s}^2} \approx 6.4\,\text{s} \tag{27}$$

Below we provide the requested plots.

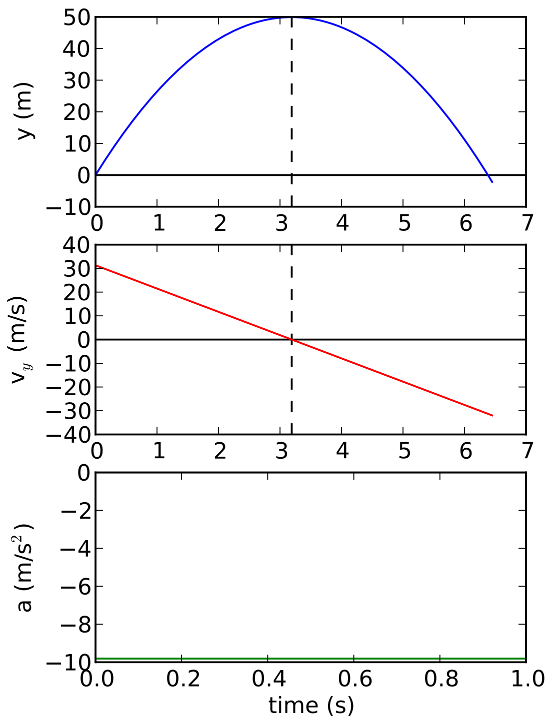


Figure 3: *Position, velocity, and acceleration for a ball thrown vertically that reaches a maximum height of* 50 m*. The vertical dashed lines show when the particle reaches maximum height and has velocity zero.*

5. Two seconds after being projected from ground level, a projectile is displaced 40 m horizontally and 53 m vertically above its launch point. What are the horizontal and vertical components of the initial velocity of the projectile?

Solution: We will skip the template for this one. We know that at time t a projectile is at position (x, y). For convenience, we will define the launch position to be the origin of our coordinate system. Presuming the y axis to be vertical, with gravitational acceleration along $-y$, we can describe the position of the projectile at any given time t:

$$y(t) = v_{iy}t - \frac{1}{2}gt^2 \tag{28}$$

$$x(t) = v_{ix}t \tag{29}$$

Here v_{ix} and v_{iy} are respectively the x and y components of the initial (launch) velocity. Given that we know $x(t)$ and $y(t)$ at the time of interest, all we need to do is solve the equations above for velocity instead of position.

$$v_{iy} = \frac{y + \frac{1}{2}gt^2}{t} \tag{30}$$

$$v_{ix} = \frac{x}{t} \tag{31}$$

Given the position of the particle is $(x, y) = (40, 53)$ at time $t = 2.0\,\mathrm{s}$, a numerical solution is easy:

$$v_{iy} = 36\,\mathrm{m/s} \tag{32}$$

$$v_{ix} = 20\,\mathrm{m/s} \tag{33}$$

6. A person standing at the top of a hemispherical rock of radius R kicks a ball (initially at rest on the top of the rock) to give it horizontal velocity $\vec{\mathbf{v}}_\mathbf{i}$ as shown below. What must be its minimum initial speed if the ball is never to hit the rock after it is kicked? Note this is not circular motion.

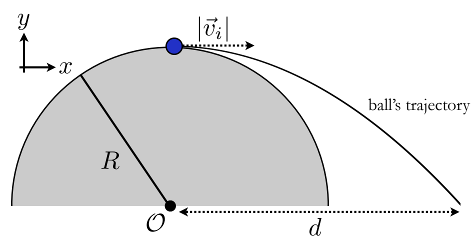


Figure 4: *A ball is kicked off the top of a rock by an unseen person.*

Solution: **Find:** The minimum speed for the ball not to hit the rock. As long as we're at it, we will also find the net horizontal distance it lands from the rock at that speed. Since the rock may be described by a circle, and the ball's motion a parabola, we are seeking a condition on the initial velocity such that the parabola always lies above the circle.

Given: The geometry of the rock, the ball's initial velocity.

Sketch: Let the x axis run horizontally and the y axis vertically, with the origin at the rock's center. This makes the ball's starting position $(0, R)$ and its launch angle with respect to the x axis $\theta = 0$ (as shown in the figure included in the problem).

Relevant equations: We need the equation of a circle of radius R centered on the origin, and the trajectory of a projectile fired at angle $\theta = 0$ relative to the x axis with starting vertical position $y(0) = R$. Let the circle be described by $y_p(x)$ and the rock $y_r(x)$. Since our solution is restricted to the upper right quadrant, the rock may be described by

$$y_r(x) = \sqrt{R^2 - x^2} \tag{34}$$

The ball's trajectory is our well-known result

$$y_p(x) = y(0) + (\tan\theta)\, x - \frac{gx^2}{2|\vec{\mathbf{v}}|^2\cos^2\theta} = R - \frac{gx^2}{2|\vec{\mathbf{v}}|^2} \tag{35}$$

Symbolic solution: The condition that the ball does not hit the rock is simply that the parabola and circle above have no intersection point, other than the trivial one at $(0, R)$. That is, the parabola must lie above the circle everywhere except $(0, R)$. Thus,

$$\begin{aligned} y_p(x) &\geq y_r(x) \\ R - \frac{gx^2}{2|\vec{\mathbf{v}}|^2} &\geq \sqrt{R^2 - x^2} \end{aligned} \tag{36}$$

In principle, this is it. Much algebra now ensues. First, simply square both sides and simplify. Since both sides must be positive for all x considered, by the problem's construction, this does not alter the inequality.

$$\begin{aligned} \left(R - \frac{gx^2}{2|\vec{\mathbf{v}}|^2}\right)^2 &\geq \left(\sqrt{R^2 - x^2}\right)^2 \\ R^2 - \frac{gRx^2}{|\vec{\mathbf{v}}|^2} + \frac{g^2x^4}{4|\vec{\mathbf{v}}|^4} &\geq R^2 - x^2 \\ \left(\frac{g^2}{4|\vec{\mathbf{v}}|^2}\right)x^4 + \left(1 - \frac{gR}{|\vec{\mathbf{v}}|^2}\right)x^2 \geq 0 & \\ x^2\left(\frac{g^2}{4|\vec{\mathbf{v}}|^2}x^2 + 1 - \frac{gR}{|\vec{\mathbf{v}}|^2}\right) \geq 0 & \\ x^2\left(\frac{g^2}{4|\vec{\mathbf{v}}|^2}x^2 + 1 - \frac{gR}{|\vec{\mathbf{v}}|^2}\right) \geq 0 & \qquad (x \neq 0) \\ \frac{g^2}{4|\vec{\mathbf{v}}|^2}x^2 + 1 - \frac{gR}{|\vec{\mathbf{v}}|^2} \geq 0 & \\ \frac{g}{4|\vec{\mathbf{v}}|^2}x^2 &\geq \left(\frac{gR}{|\vec{\mathbf{v}}|^2} - 1\right) \end{aligned} \tag{37}$$

We require this inequality to be true for all $x > 0$ for the ball not to hit the rock anywhere in the domain of interest. The only way this can happen is if the right-hand side is negative:

$$\begin{aligned} \frac{gR}{|\vec{\mathbf{v}}|^2} - 1 \leq 0 & \\ \implies \quad |\vec{\mathbf{v}}| \geq \sqrt{gR} & \end{aligned} \tag{38}$$

Note that this is *not* the same condition you would find by simply requiring the particle's *range* to be larger than R. It is easy to verify that one can make a parabola with horizontal range R in this situation that still intersects the circle . . . try it out!

Where does the projectile land? Clearly, at $y_p = 0$, since that is where the ground is. We simply need to set the ball's y position equal to zero, and solve for the resulting x value using our minimal velocity from Eq. 38. This is where the ball lands.

$$\begin{aligned} y_p = 0 &= R - \frac{gx^2}{2|\vec{\mathbf{v}}|^2} = R - \frac{gx^2}{2gR} = R - \frac{x^2}{2R} \\ 2R^2 &= x^2 \\ \implies \quad x &= R\sqrt{2} \end{aligned} \tag{39}$$

What is more interesting is *how far from the rock* the ball lands. Since the rock extends to $x = R$, we have gone beyond that by a distance

$$\text{distance from rock} = R\left(\sqrt{2} - 1\right) \tag{40}$$

Numeric solution: Numbers? How awkward. $\sqrt{2} \approx 1.41$, $\sqrt{2} - 1 \approx 0.41$. The ball lands about 40% of the rock's radius beyond its base. With $g \approx 10$, and $\sqrt{g} \approx 3.2$, the maximal velocity is about $3.2\sqrt{R}$.

Double check: Things you can do: simply graph the two trajectories you came up with for a given value of R, and verify they do not intersect. Check the units of the final answer. Check that the ball lands beyond the base of the rock (it does).

Another way: Since the parabola has a maximal radius of curvature at at its apex, with a little geometrical reasoning you can prove that if the circle and parabola are tangent at the parabola's apex, and the parabola's radius of curvature there exceeds R, the two curves cannot intersect. It does work: calculate the parabola's radius of curvature, insist that it be larger than R, and the same condition results: $v \geq \sqrt{gR}$. I don't really have the stamina to work up a full geometric proof of that, however . . . perhaps one of you would do it for extra credit?

7. Here are three vectors:

$$\begin{aligned} \vec{\mathbf{d}}_1 &= -2.0\,\hat{\imath} + 3.0\,\hat{\jmath} + 2.0\,\hat{\mathbf{k}} \\ \vec{\mathbf{d}}_2 &= -3.0\,\hat{\imath} - 4.0\,\hat{\jmath} - 2.0\,\hat{\mathbf{k}} \\ \vec{\mathbf{d}}_3 &= 1.0\,\hat{\imath} + 3.0\,\hat{\jmath} + 5.0\,\hat{\mathbf{k}} \end{aligned}$$

What is the result of the following operations?

a) $\vec{\mathbf{d}}_1 \cdot \left(\vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3\right)$
b) $\vec{\mathbf{d}}_1 \cdot \left(\vec{\mathbf{d}}_2 \times \vec{\mathbf{d}}_3\right)$
c) $\vec{\mathbf{d}}_1 \times \left(\vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3\right)$

Solution: **Given:** Three vectors $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$, and $\vec{\mathbf{d}}_3$ in two-dimensional cartesian coordinates.

Find: The result of various sums and scalar and vector products given above.

Sketch: Not really necessary.

Relevant equations: In this case, we need only the requisite formulas for adding two vectors and taking the scalar and vector products of two vectors. Given two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$,

$$\vec{\mathbf{a}} = a_x\,\hat{\boldsymbol{\imath}} + a_y\,\hat{\boldsymbol{\jmath}} + a_z\,\hat{\mathbf{k}}$$
$$\vec{\mathbf{b}} = b_x\,\hat{\boldsymbol{\imath}} + b_y\,\hat{\boldsymbol{\jmath}} + b_z\,\hat{\mathbf{k}}$$

Then

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = (a_x + b_x)\ \hat{\boldsymbol{\imath}} + (a_y + b_y)\ \hat{\boldsymbol{\jmath}} + (a_z + b_z)\ \hat{\mathbf{k}}$$
$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z$$
$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \det \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\,\hat{\mathbf{x}} + (a_z b_x - a_x b_z)\,\hat{\mathbf{y}} + (a_x b_y - a_y b_x)\,\hat{\mathbf{z}}$$

The only other thing we need remember is to work first inside the parenthesis. For example, for the first quantity, we need to find $\vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3$ first, and then calculate the scalar product of it with $\vec{\mathbf{d}}_1$.

Symbolic solution:
We can first find the results in a purely symbolic fashion by defining

$$\vec{\mathbf{d}}_1 = -2.0\,\hat{\boldsymbol{\imath}} + 3.0\,\hat{\boldsymbol{\jmath}} + 2.0\,\hat{\mathbf{k}} = d_{1x}\,\hat{\boldsymbol{\imath}} + d_{1y}\,\hat{\boldsymbol{\jmath}} + d_{1z}\,\hat{\mathbf{k}}$$

and similarly for $\vec{\mathbf{d}}_2$ and $\vec{\mathbf{d}}_3$. Finding the answer symbolically first has the advantage of being more amenable to double-checking our work later on . . . though it will require a bit more algebra in the intermediate steps. So it goes.

Starting with the first quantity, working first inside the parenthesis:

$$\begin{aligned}\vec{\mathbf{d}}_1\cdot\left(\vec{\mathbf{d}}_2+\vec{\mathbf{d}}_3\right) &= \vec{\mathbf{d}}_1\cdot\left[\left(d_{2x}+d_{3x}\right)\,\hat{\boldsymbol{\imath}}+\left(d_{2y}+d_{3y}\right)\,\hat{\boldsymbol{\jmath}}+\left(d_{2z}+d_{3z}\right)\,\hat{\mathbf{k}}\right]\\ &= \left[d_{1x}\,\hat{\boldsymbol{\imath}}+d_{1y}\,\hat{\boldsymbol{\jmath}}+d_{1z}\,\hat{\mathbf{k}}\right]\cdot\left[\left(d_{2x}+d_{3x}\right)\,\hat{\boldsymbol{\imath}}+\left(d_{2y}+d_{3y}\right)\,\hat{\boldsymbol{\jmath}}+\left(d_{2z}+d_{3z}\right)\,\hat{\mathbf{k}}\right]\\ &= d_{1x}\left(d_{2x}+d_{3x}\right)+d_{1y}\left(d_{2y}+d_{3y}\right)+d_{1z}\left(d_{2z}+d_{3z}\right)\end{aligned}$$

For the second quantity, we first need to calculate the cross product of the second and third vectors. It is a bit messy, but brute force is really the only way forward.

$$\begin{aligned}\vec{\mathbf{d}}_1\cdot\left(\vec{\mathbf{d}}_2\times\vec{\mathbf{d}}_3\right) &= \vec{\mathbf{d}}_1\cdot\left[\left(d_{2y}d_{3z}-d_{2z}d_{3y}\right)\,\hat{\boldsymbol{\imath}}+\left(d_{2z}d_{3x}-d_{2x}d_{3z}\right)\,\hat{\boldsymbol{\jmath}}+\left(d_{2x}d_{3y}-d_{2y}d_{3x}\right)\,\hat{\mathbf{k}}\right]\\ &= \left[d_{1x}\,\hat{\boldsymbol{\imath}}+d_{1y}\,\hat{\boldsymbol{\jmath}}+d_{1z}\,\hat{\mathbf{k}}\right]\cdot\left[\left(d_{2y}d_{3z}-d_{2z}d_{3y}\right)\,\hat{\boldsymbol{\imath}}+\left(d_{2z}d_{3x}-d_{2x}d_{3z}\right)\,\hat{\boldsymbol{\jmath}}+\left(d_{2x}d_{3y}-d_{2y}d_{3x}\right)\,\hat{\mathbf{k}}\right]\\ &= d_{1x}\left(d_{2y}d_{3z}-d_{2z}d_{3y}\right)+d_{1y}\left(d_{2z}d_{3x}-d_{2x}d_{3z}\right)+d_{1z}\left(d_{2x}d_{3y}-d_{2y}d_{3x}\right)\end{aligned}$$

The third quantity is no more difficult; this time we first perform the addition, and then take a cross product:

$$\begin{aligned}\vec{\mathbf{d}}_1\times\left(\vec{\mathbf{d}}_2+\vec{\mathbf{d}}_3\right) &= \vec{\mathbf{d}}_1\times\left[\left(d_{2x}+d_{3x}\right)\,\hat{\boldsymbol{\imath}}+\left(d_{2y}+d_{3y}\right)\,\hat{\boldsymbol{\jmath}}+\left(d_{2z}+d_{3z}\right)\,\hat{\mathbf{k}}\right]\\ &= \left[d_{1y}\left(d_{2z}+d_{3z}\right)-d_{1z}\left(d_{2y}+d_{3y}\right)\right]\,\hat{\boldsymbol{\imath}}+\left[d_{1z}\left(d_{2x}+d_{3x}\right)-d_{1x}\left(d_{2z}+d_{3z}\right)\right]\,\hat{\boldsymbol{\jmath}}\\ &\qquad+\left[d_{1x}\left(d_{2y}+d_{3y}\right)-d_{1y}\left(d_{2x}+d_{3x}\right)\right]\,\hat{\mathbf{k}}\end{aligned}$$

There is not much point in simplifying further, there are no like terms to collect.

Numeric solution:
All that is needed now is to plug in the actual numbers, noting that $d_{1x}=-3.0$, $d_{1y}=3.0$, $d_{1z}=2.0$, *etc.* For the first quantity:

$$\begin{aligned}\vec{\mathbf{d}}_1\cdot\left(\vec{\mathbf{d}}_2+\vec{\mathbf{d}}_3\right) &= d_{1x}\left(d_{2x}+d_{3x}\right)+d_{1y}\left(d_{2y}+d_{3y}\right)+d_{1z}\left(d_{2z}+d_{3z}\right)\\ &= -2.0\left(-3.0+1.0\right)+3.0\left(-4.0+3.0\right)+2.0\left(-2.0+5.0\right)=4.0-3.0+6.0=7.0\end{aligned}$$

For the second quantity:

$$\begin{aligned}\vec{\mathbf{d}}_1\cdot\left(\vec{\mathbf{d}}_2\times\vec{\mathbf{d}}_3\right) &= d_{1x}\left(d_{2y}d_{3z}-d_{2z}d_{3y}\right)+d_{1y}\left(d_{2z}d_{3x}-d_{2x}d_{3z}\right)+d_{1z}\left(d_{2x}d_{3y}-d_{2y}d_{3x}\right)\\ &= -2.0\left(-20.0+6.0\right)+3.0\left(-2.0+15.0\right)+2.0\left(-9.0+4.0\right)=28.0+39.0-10=57\end{aligned}$$

For the third quantity:

$$
\begin{aligned}
&[d_{1y}(d_{2z}+d_{3z}) - d_{1z}(d_{2y}+d_{3y})]\,\hat{\boldsymbol{\imath}} + [d_{1z}(d_{2x}+d_{3x}) - d_{1x}(d_{2z}+d_{3z})]\,\hat{\boldsymbol{\jmath}} \\
&\qquad\qquad\qquad\qquad + [d_{1x}(d_{2y}+d_{3y}) - d_{1y}(d_{2x}+d_{3x})]\,\hat{\mathbf{k}} \\
&= [3.0(-2.0+5.0) - 2.0(-4.0+3.0)]\,\hat{\boldsymbol{\imath}} + [2.0(-3.0+1.0) + 2.0(-2.0+5.0)]\,\hat{\boldsymbol{\jmath}} \\
&\qquad\qquad\qquad\qquad + [-2.0(-4.0+3.0) - 3.0(-3.0+1.0)]\,\hat{\mathbf{k}} \\
&= [9.0+2.0]\,\hat{\boldsymbol{\imath}} + [-4+6.0]\,\hat{\boldsymbol{\jmath}} + [2.0+6.0]\,\hat{\mathbf{k}} \\
&= 11.0\,\hat{\boldsymbol{\imath}} + 2.0\,\hat{\boldsymbol{\jmath}} + 8.0\,\hat{\mathbf{k}}
\end{aligned}
$$

Double check: *Units. Order-of-magnitude.* There are no units in this problem, but we can decide what sort of solution should we expect *qualitatively* – should the answers be vectors, scalars, or neither?

For the first quantity, the quantity inside parenthesis is the sum of two vectors, and therefore a vector itself. We then need to find the scalar product of this vector with $\vec{\mathbf{d}}_1$. The final quantity, then, is the scalar product of two vectors, which is itself a scalar (*i.e.,* just a number). This also means that the product should have only terms with the product of two components, such as $d_{1x}d_{2x}$, which is consistent with our answer.

The second quantity is similarly a scalar, since the cross product in parenthesis results in an (axial) vector, whose scalar product with $\vec{\mathbf{d}}_1$ also gives a scalar. Since there are two products here, the final answer should have only terms with three components, such as $d_{1x}d_{2y}d_{3z}$, consistent with our answer.

The third quantity has a vector resulting in the parenthesis, and we need its vector product with $\vec{\mathbf{d}}_1$, which gives an (axial) vector. Thus, only the third quantity is a vector at all, and only a pseudovector at that, the other two are just numbers. Again, we have only one product here, so the final answer should again have terms with two components.

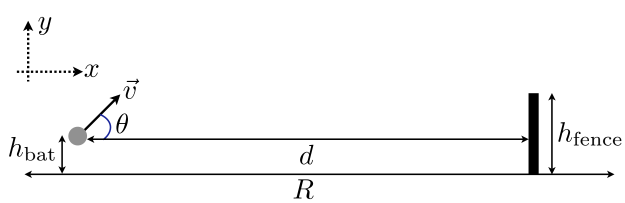
8. A batter hits a baseball coming off of the bat at a 45° angle, making contact a distance 1.22 m above the ground. Over level ground, the batted ball has a range of 107 m. Will the ball clear a 7.32 m tall fence at a distance of 97.5 m? Justify your answer. *Hint: use the range equation to get the velocity, then use the trajectory equation to find the path of the ball.*

Solution: This is problem 4.47 from your textbook.

Find: Whether a batted baseball clears a fence, and by what amount it does or does not.

Given: The baseball's initial launch height and angle, the range the baseball would have without the fence, the distance to the fence and its height.

Sketch: Let the y axis run vertically and the x axis horizontally as shown below. Let the range the baseball would have without the fence be $R = 107\,\mathrm{m}$, with the distance to the fence $d = 97.5\,\mathrm{m}$ and its height $h_{\text{fence}} = 7.32\,\mathrm{m}$. The baseball is batted at an angle $\theta = 45°$ at speed v_i a height of $h_{\text{bat}} = 1.22\,\mathrm{m}$ above the ground.



Let the origin be at the position the ball leaves the bat. The height of the fence *relative to the height of the bat* is then

$$\delta h = h_{\text{fence}} - h_{\text{bat}}$$

What we really need to determine is the ball's y coordinate at $x = d$. If $y > \delta h$, the ball clears the fence. We can use the range the baseball would have without the fence and the launch angle to find the ball's speed, which will allow a complete calculation of the trajectory.

Relevant equations: We need only the equations for the range and trajectory of a projectile over level ground:

$$R = \frac{v_i^2 \sin 2\theta}{g}$$

$$y(x) = x \tan\theta - \frac{gx^2}{2v_i^2 \cos^2\theta}$$

Symbolic solution: From the range equation above, we can write the velocity in terms of known quantities:

$$v_i = \sqrt{\frac{Rg}{\sin 2\theta}}$$

The trajectory then becomes

$$y(x) = x \tan\theta - \frac{gx^2 \sin 2\theta}{2Rg\cos^2\theta} = x\tan\theta - \frac{x^2 \sin 2\theta}{2R\cos^2\theta}$$

The height difference between the ball and the fence is $y(d) - \delta h$. If it is positive, the ball clears the fence.

$$\begin{aligned} \text{clearance} = y(d) - \delta h = d\tan\theta - \frac{d^2 \sin 2\theta}{2R\cos^2\theta} - \delta h = d\tan\theta - \frac{d^2\sin 2\theta}{2R\cos^2\theta} - h_{\text{fence}} + h_{\text{bat}} \\ = d\tan\theta - \frac{2d^2 \sin\theta\cos\theta}{2R\cos^2\theta} - h_{\text{fence}} + h_{\text{bat}} = d\tan\theta\left(1 - \frac{d}{R}\right) - h_{\text{fence}} + h_{\text{bat}} \end{aligned}$$

Numeric solution: Using the numbers given, and noting $\tan 45° = 1$, $\sin 90° = 1$, and $\cos^2 45° = 1/2$

$$\text{clearance} \approx 2.56\,\text{m}$$

The ball does clear the fence, by approximately 2.56 m.

9. A ball rolls horizontally off the top of a stairway with a speed of 1.52 m/s. The steps are 20.3 cm high and 20.3 cm wide. Which step does the ball hit first. You may assume that there are many, many stairs.

Solution: **Given:** The dimensions of a staircase, and the initial velocity of a ball which rolls off the staircase. Let the staircase width and height be d, and the ball's initial speed $|\vec{\mathbf{v}}_i|$.

Find: Which step the ball first hits on its way down.

Sketch: We choose a cartesian coordinate system with x and y axes aligned with the stairs, as shown below. Let the origin be the point at which the ball leaves the topmost stair. The ball is launched horizontally off of the top step, and will follow a parabolic trajectory down the staircase. How do we determine which stair will first be hit? From the sketch, it is clear that we need to find at which point the ball's parabolic trajectory (solid curve) passes below a line connecting the right-most tips of each stair (dotted line).

Relevant equations: Based on our logic above, we need an equation for the ball's trajectory and an equation for the line describing the staircase boundary. The staircase itself is is composed of steps of equal height and width. Therefore, a line from the origin connecting the right-most tip of each stair (the dotted line in the figure) will have a slope of -1, and can be described by $y_s = -x$.

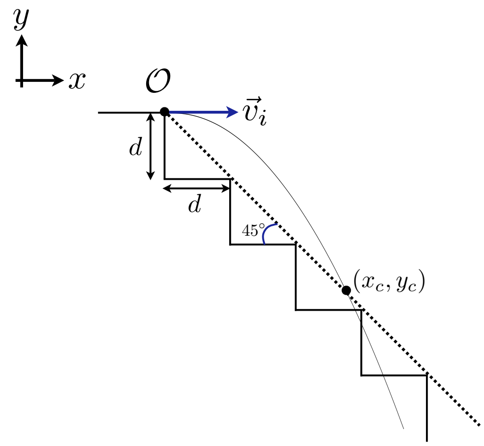


Figure 5: *A projectile launched horizontally off the top of a staircase.*

The ball will follow our now well-known parabolic trajectory. In this case, the launch angle is zero, and the ball's motion is described by setting $\theta = 0$ in Eq. **??**:

$$y_b = -\frac{gx^2}{2|\vec{\mathbf{v}}_i|^2} \tag{41}$$

We first need to find the x coordinate where $y_b = y_x$, which is the point where the parabolic trajectory dips below the line defining the staircase slope. Call this coordinate x_c. Given this coordinate, we need to determine how many stairs this distance corresponds to. The ratio of x_c to the stair width should give us this number. However, we must keep in mind the fact that the staircase is discrete: if we find that x_c corresponds to, for example, 4.7 stair widths, what does that mean? It means the ball crossed the fourth stair, but 70% of the way across the fifth one, its trajectory dipped below the line defining the staircase. Thus, the ball would hit the fifth stair.

What we need, then, is to *find the ratio of x_c and the stair width d, and take the next largest integer.* This gives us the number of the stair the ball first hits n_s. There is a mathematical function that does exactly what we want, for this operation, the *ceiling* function. It takes a real-valued argument x and gives back the next-highest integer. For example, if $x = 3.2$, then the ceiling of x is 4. The standard notation is $\lceil x \rceil = 4$.[i].

$$n_s = \left\lceil \frac{x_c}{d} \right\rceil$$

Symbolic solution: First, we need to find the point x_c where the ball's parabolic trajectory intersects the staircase boundary line:

[i]If you want to get all technical, $\lceil x \rceil = \min \{n \in \mathbb{Z} \,|\, n \geq x\}$

$$
\begin{aligned}
y_b = -\frac{gx^2}{2|v_i|^2} &= y_s = -x \\
0 &= \frac{gx^2}{2|\vec{\mathbf{v}}_i|^2} - x \\
0 &= x\left[\frac{gx}{2|\vec{\mathbf{v}}_i|^2} + 1\right] \\
\Longrightarrow \quad x_c &= \left\{0, \frac{2|\vec{\mathbf{v}}_i|^2}{gd}\right\}
\end{aligned}
$$

As usual, one of our answers is the trivial solution, the one where the ball never leaves the staircase ($x_c\!=\!0$). The second solution is what we are after. The number of the stair that is first hit is then

$$n_s = \left\lceil \frac{2|\vec{\mathbf{v}}_i|^2}{gd} \right\rceil$$

Numeric solution: We are given $|\vec{\mathbf{v}}_i|^2 = 1.52\,\mathrm{m/s}$ and $d = 20.3\,\mathrm{cm} = 0.203\,\mathrm{m}$. Additionally, we need $g\!\approx\!9.81\,\mathrm{m/s^2}$.

$$n_s = \left\lceil \frac{2|\vec{\mathbf{v}}_i|^2}{gd} \right\rceil = \left\lceil \frac{2\,(1.52\,\mathrm{m/s})^2}{(9.81\,\mathrm{m/s^2})\,(0.203\,\mathrm{m})} \right\rceil = \left\lceil 2.32 \right\rceil = 3$$

The ball will hit the third stair.

Double check: The ratio x_c/d must be dimensionless, as we have shown it to be above; our units are correct. Another "brute force" method of checking our result is to calculate the y position of the projectile at the right-most edge of successive stairs. At the right-most edge of the n^{th} stair, the x coordinate is nd. If the y coordinate of the projectile's trajectory is *below* $(-nd)$ for the right-most edge of a given stair, then we must have hit that stair.

Stair n	$x\!=\!nd$ **(m)**	$y_s\!=\!nd$ **(m)**	y_b **(m)**	**result**
1	0.203	-0.203	-0.0875	cleared
2	0.406	-0.406	-0.350	cleared
3	0.609	-0.609	-0.787	hit

The brute-force method confirms our result: the third stair is not cleared. While arguably faster, this method lacks a certain . . . elegance. It is fine for double-checking, but purely symbolic solutions are *always* preferred when they are possible.

10. A projectile's launch speed is five times its speed at maximum height. Find the launch angle θ_o.

Solution: Presume the vertical direction to be the y axis. At maximum height, we know the vertical component of the velocity is instantaneously zero, $v_y = 0$. The horizontal component of the velocity v_x remains constant throughout the motion. Thus, at maximum height the velocity is purely in the horizontal x direction and has magnitude v_x. Given an initial velocity vector $\vec{\mathbf{v}}_o$ and launch angle θ_o, we know $v_x = |\vec{\mathbf{v}}_o| \cos\theta_o$.

We are told that the launch speed $|\vec{\mathbf{v}}_o|$ is five times the the speed at maximum height. At maximum height the speed is $|\vec{\mathbf{v}}_o| \cos\theta_o$, thus,

$$|\vec{\mathbf{v}}_o| = 5|\vec{\mathbf{v}}_o| \cos\theta_o \tag{42}$$

$$\implies \qquad \cos\theta_o = \frac{1}{5} \tag{43}$$

$$\theta \approx 78.5^\circ \tag{44}$$

Appendix: making a simple plot in Python

I made the plots above using simple Python code. Obviously one could just use Excel or something, and for small tasks like this, it would be faster. However, we will have occasion to do some numerical simulations as the semester progresses, and this will be a handy trick. The code could be more elegant, and the resulting plots much prettier, but I aimed for simplicity.

```
import numpy as np
import matplotlib.pyplot as plt

#define velocity, position, and acceleration functions
def x(t):
    return 20*t-5*t*t*t

def v(t):
        return 20-15*t*t

def a(t):
        return -30*t

#specify a range of time for the plot
t = np.arange(-2.5, 2.5, 0.05)

# one sub-plot for x(t). stack all subplots vertically
# we have 3 rows, 1 column, this is plot 1
#y label, lines for y=0 and x=0
plt.subplot(311)
plt.ylabel('x (m)')
plt.axhline(y=0,c='k')
plt.axvline(x=0,c='k')
plt.plot(t, x(t),c='b')

# one for v(t)
plt.subplot(312)
plt.ylabel('v (m/s)')
plt.axhline(y=0,c='k')
plt.axvline(x=0,c='k')
plt.plot(t, v(t), c='r')

#one for a(t)
plt.subplot(313)
plt.ylabel('a (m/s$^{2}$)')
plt.axhline(y=0,c='k')
plt.axvline(x=0,c='k')
plt.plot(t, a(t), c='k')

plt.xlabel('time (s)')  #shared label on x axis

plt.subplots_adjust(left=0.1, right=0.5, top=0.9, bottom=0.1)

#plt.show()  #write to screen
plt.savefig('foo.pdf',bbox_inches='tight') #write to file
```