Problem Set 5 Solutions

1. A bowler throws a bowling ball of radius R along a lane. The ball slides on the lane with initial speed v_o and initial angular speed $\omega_o = 0$. The coefficient of kinetic friction between the ball and the lane is μ_k. The kinetic frictional force $\vec{f}_k$ acting on the ball causes a linear acceleration of the ball while producing a torque that causes an angular acceleration of the ball. When the center of mass speed v_{cm} has decreased enough and angular speed ω has increased enough, the ball stops sliding and then rolls smoothly. **(a)** What then is the center of mass speed v_{cm} in terms of ω? During the sliding, what are the ball's **(b)** linear acceleration and **(c)** angular acceleration? **(d)** How long does the ball slide? **(e)** How far does the ball slide? **(f)** What is the linear speed of the ball when smooth rolling begins?

Solution: **(a)** Initially, the bowling ball is purely sliding, and as friction takes hold, the ball begins to roll. During the pure sliding phase, the ball rotates about its center of mass, independent of the overall center of mass motion.
After sufficient time, the rolling motion "catches up" with the sliding motion, and the ball begins to roll - it is no longer spinning about its center of mass, rolling smoothly. This smooth rolling is equivalent to a rotation about a point on the *surface* of the ball (not the center of mass), and as we derived earlier, this means that at the point we have smooth rolling motion, center of mass velocity and angular velocity are simply related:

$$v_{com} = r\omega$$

Here r is the given radius of the ball. During the sliding phase, we should write $v_{com} > r\omega$. The angular velocity is not high enough for the ball to "catch" on the lane. [i]

(b) During the sliding phase, rotation is irrelevant to the dynamics - it is just like any other sliding object we have analyzed. A force of kinetic friction acts at the interface between the ball and the lane, which is equal in magnitude to $f_k = \mu_k F_N$, where μ_k is the coefficient of kinetic friction and $F_N = mg$ the normal force. Since this is the only force acting, we can easily apply Newton's law:

$$\sum F = ma = -f_k$$
$$a = -f_k/m = -\mu_k g$$

(c) The angular acceleration α during the sliding phase is also provided by the friction force f_k. The friction force acts at a distance r from the center of mass, and at a right angle to a radius drawn from the center of mass to the intersection between the ball and lane. Thus, f_k also provides a torque τ, and as the only torque present, it must equal the moment of inertia of the ball times the angular acceleration. Noting $I = \frac{2}{5}mr^2$ for a solid sphere,

$$\tau_{\text{net}} = r f_k = I\alpha = \frac{2}{5}mr^2\alpha$$
$$\alpha = \frac{r f_k}{\frac{2}{5}mr^2} = \frac{5\mu_k mg}{2mr} = \frac{5\mu_k g}{2r}$$

---

[i]My parents used to own a bowling alley. I can go into much more detail on this problem for the curious.

(d) During the sliding phase, the rotational and translational motion are essentially decoupled, and we can consider the center of mass motion from the point of view of standard kinematics. That is,

$$v_{com}(t) = v_{com}(0) + at = v_{com}(0) - \mu_k g t$$

Here $v_{com}(0)$ is the initial center of mass velocity, and we imply $t=0$ at the moment the ball hits the lane. The same is true for the rotational motion, with the added simplification that the initial angular velocity is zero:

$$\omega(t) = \omega(0) + \alpha t = \left(\frac{5\mu_k g}{2r} \right) t$$

Say that the sliding stops at a time t_o. At the moment that sliding stops, we know that $v_{com}(t_o) = r\omega(t_o)$. This yields t_o, the time it takes to stop sliding, in terms of known quantities:

$$v_{com}(t_o) = r\omega(t_o)$$
$$v_{com}(0) - \mu_k g t_o = \frac{5}{2}\mu_k g t_o$$
$$t_o \left(\frac{5}{2}\mu_k g + \mu_k g \right) = v_{com}(0)$$
$$t_o = \frac{v_{com}(0)}{\frac{7}{2}\mu_k g} = \frac{2v_{com}(0)}{7\mu_k g}$$

(e) Given the time to stop sliding, we can also find the distance covered d by standard kinematics:

$$d = v_{com}(0)t_o + \frac{1}{2}at_o^2$$
$$= v_{com}(0) \left(\frac{2v_{com}(0)}{7\mu_k g} \right) - \frac{1}{2}\mu_k g \left(\frac{2v_{com}(0)}{7\mu_k g} \right)^2$$
$$= \frac{2[v_{com}(0)]^2}{7\mu_k g} - \frac{2[v_{com}(0)]^2}{49\mu_k g}$$
$$= \frac{12[v_{com}(0)]^2}{49\mu_k g}$$

(f) The linear (i.e., center of mass) speed at the moment sliding stops is also just kinematics:

$$v_{com}(t_o) = v_{com}(0) + at = v_{com}(0) - \mu_k g t$$

2. In the figure below, a small block of mass m slides down a frictionless surface through height h and then sticks to a uniform rod of mass M and length L. The rod pivots about point O through angle θ before momentarily stopping. Find θ.

Solution: **Solution:** Referring to the sketch above, let A be the starting point, B the moment of collision between the ball and rod, and C the point when maximum height is reached by the rod + ball system. We approximate the ball as a point mass, since we are told it is small (and we anyway have no way of calculating its moment of inertia, since we do not have any geometrical details ...).
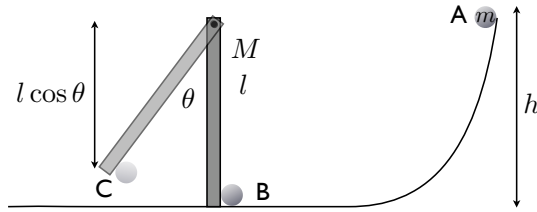
The velocity v of the ball at point B can be found using conservation of mechanical energy. Let the floor be the height of zero gravitational potential energy.

$$K_A + U_A = K_B + U_B$$
$$mgh = \frac{1}{2}mv^2$$
$$\implies v = \sqrt{2gh}$$

The collision is clearly inelastic, since the ball sticks to the rod. We could use conservation of linear momentum, but this would require breaking up the rod into infinitesimal discrete bits of mass and integrating over its length. Easier is to use conservation of *angular* momentum about the pivot point of the rod. Just before the collision, we have the ball moving at speed v a distance l. Let $\hat{\imath}$ be to the right, and $\hat{\jmath}$ upward (making $\hat{\mathbf{k}}$ into the page). The initial angular momentum is then

$$\vec{\mathbf{L}}_i = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = l\,\hat{\jmath} \times (-mv\,\hat{\imath}) = -mvl\,(\hat{\jmath} \times \hat{\imath}) = mvl\,\hat{\mathbf{k}} = ml\sqrt{2gh}\,\hat{\mathbf{k}}$$

After the collision, we have the rod and mass stuck together, rotating at angular velocity ω. Defining counterclockwise rotation to be positive as usual, the final angular momentum is thus

$$\vec{\mathbf{L}}_f = I\omega\,\hat{\mathbf{k}}$$

The total moment of inertia about the pivot point is that of the rod rotating plus that of the ball. The rod rotates a distance $l/2$ from its center of mass, and again we approximate the ball as a point mass rotating at a distance l (since we told it is small).

$$I = I_{\text{rod}} + I_{\text{ball}} = I_{\text{rod, com}} + M\left(\frac{l}{2}\right)^2 + ml^2 = \frac{1}{12}Ml^2 + Ml^2 + ml^2 = \left(\frac{1}{3}M + m\right)l^2$$

Equating initial and final angular momentum, we can solve for the angular velocity after the collision:

$$L_f = I\omega = L_i = mvl = ml\sqrt{2gh}$$
$$\left(\frac{1}{3}M + m\right)l^2\omega = ml\sqrt{2gh}$$
$$\omega = \frac{m\sqrt{2gh}}{\left(\frac{1}{3}M + m\right)l}$$

At this point, we may use conservation of energy once again. When the system reaches its maximum angle θ at C, the center of mass of the rod + ball system will have moved up by an amount Δy_{cm}. The change in gravitational potential energy related to this change in center of mass height must be equal to the rotational kinetic energy just after the collision. Thus,

$$\frac{1}{2}I\omega^2 = \frac{\vec{\mathbf{L}} \cdot \vec{\mathbf{L}}}{2I} = \frac{L^2}{2I} = (m + M)\, g \Delta y_{cm}$$

Here we have noted that the rotational kinetic energy can be related to the angular momentum to save a bit of algebra. To proceed, we must find the difference in the center of mass height between points C and B. Let $y=0$ be the height of the floor. At point B,

$$y_{cm,B} = \frac{M\left(\dfrac{L}{2}\right) + m\,(0)}{m + M} = \left(\frac{l}{2}\right)\left(\frac{M}{m + M}\right)$$

At point C, the ball is now at a height $l - l\cos\theta$, while the center of mass of the rod (its midpoint) is now at $l - l\cos\theta + \frac{1}{2}l\cos\theta$. Thus,

$$y_{cm,C} = \frac{M\left(l - l\cos\theta + \dfrac{1}{2}l\cos\theta\right) + m\,(l - l\cos\theta)}{m + M} = \frac{Ml\left(1 - \dfrac{1}{2}\cos\theta\right) + ml\,(1 - \cos\theta)}{m + M}$$

The change in center of mass height can now be found:

$$\Delta y_{cm} = y_{cm,C} - y_{cm,B} = \frac{Ml\left(1 - \dfrac{1}{2}\cos\theta\right) + ml\,(1 - \cos\theta) - \dfrac{1}{2}Ml}{m + M}$$

$$= \frac{\dfrac{1}{2}Ml\,(1 - \cos\theta) + ml\,(1 - \cos\theta)}{m + M}$$

$$= \frac{l}{m + M}\,(1 - \cos\theta)\left(m + \frac{1}{2}M\right)$$

Using our previous energy balance between B and C,

$$\frac{L^2}{2I} = (m + M)\, g \Delta y_{cm} = lg\,(1 - \cos\theta)\left(m + \frac{1}{2}M\right)$$

We could also have found the change in potential energy a bit more easily by just separately considering the change energy due to the change in height of the center of mass of the rod and the ball separately and adding the two together. The putty changes height by $l - l\cos\theta$, while the rod's center of mass changes height by half that much.

$$\Delta U_{\text{ball}} = mgh_{\text{ball}} = mgl\,(1 - \cos\theta) \tag{1}$$

$$\Delta U_{\text{rod}} = Mg_{\text{rod, cm}} = Mg\frac{l}{2}\,(1 - \cos\theta) \tag{2}$$

$$\Delta U_{\text{tot}} = \Delta U_{\text{ball}} + \Delta U_{\text{rod}} = gl\,(1 - \cos\theta)\left(m + \frac{1}{2}M\right) \tag{3}$$

Since the initial and final angular momenta are equal, we may substitute either L_f or L_i, the latter being the easiest option. This is not strictly *necessary* – we could use L_f or even just grind through $\frac{1}{2}I\omega^2$ and the result must be the same. However, using L_i here saves quite a bit of algebra in the end when we try to put θ in terms of only given quantities. Doing so, and solving for θ

$$\frac{L_f^2}{2I} = \frac{L_i^2}{2I} = \frac{2l^2 m^2 gh}{2\left(\frac{1}{3}M + m\right)l^2} = lg\left(1 - \cos\theta\right)\left(m + \frac{1}{2}M\right)$$

$$1 - \cos\theta = \frac{m^2 h}{l\left(\frac{1}{3}M + m\right)\left(\frac{1}{2}M + m\right)}$$

$$\theta = \cos^{-1}\left[1 - \frac{m^2 h}{l\left(\frac{1}{3}M + m\right)\left(\frac{1}{2}M + m\right)}\right]$$

Note that for $m = 0$, $\theta = 0$, as we expect. On the other hand, for $M = 0$ we have $\cos\theta = 1 - h/l = 1/2$. This means that the particle is at a height $l - l\cos\theta = l/2 = h$ at point C – exactly what we would expect if mechanical energy were conserved!

3. In the figure below, two balls of mass m are attached to the ends of a thin rod of length L and negligible mass. The rod is free to rotate in a vertical plane without friction about a horizontal axis through its center. With the rod initially horizontal, a wad of wet clay of mass M drops onto one of the balls, hitting it with a speed of v_i and then sticking to it. **(a)** What is the angular speed of the system just after the putty wad hits? **(b)** What is the ratio of the kinetic energy of the system after the collision to that of the putty wad just before? **(c)** Through what angle will the system rotate before it momentarily stops?
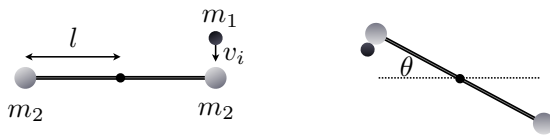
Solution: **Solution: (a)** Our dumbbell, consisting of two masses m_2 both a distance l from its center of mass, is struck by a smaller mass m_1 traveling at velocity $\vec{v}_i$. Conservation of angular momentum can be used to find the angular velocity after the collision. Before the collision, with $\hat{\imath}$ to the right and $\hat{\jmath}$ upward, we have the smaller mass' momentum $\vec{p}_i = -m_1 v_i\,\hat{\jmath}$ acting at a distance $\vec{r} = l\,\hat{\imath}$ from the center of rotation.

$$\vec{L}_i = \vec{r} \times \vec{p} = -m_1 l v_i\,\hat{k}$$

The minus sign indicates a clockwise rotation following our usual convention, which is sensible. After the collision, the entire system rotates clockwise at angular velocity $\vec{\omega} = -\omega\,\hat{k}$. The total moment of inertia is found easily, since we have only point-like masses:

$$I = \sum_i m_i r_i^2 = m_2 l^2 + m_2 l^2 + m_1 l^2 = l^2\left(2m_2 + m_1\right)$$

The final angular momentum is then

$$\vec{L}_f = I\vec{\omega} = -l^2\omega\left(2m_2 + m_1\right)\hat{\mathbf{k}}$$

Conservation of angular momentum gives us

$$\vec{\omega} = \frac{L_i}{I} = \frac{m_1 v_i}{\left(2m_2 + m_1\right)l}\hat{\mathbf{k}}$$

(b) The initial kinetic energy of the system is only that of the smaller mass, $K_i = \frac{1}{2}m_1 v_i^2$. The final kinetic energy is the rotational kinetic energy of the whole system, which is simplified a bit in terms of angular momentum

$$K_f = \frac{1}{2}I\omega^2 = \frac{\vec{\mathbf{L}}\cdot\vec{\mathbf{L}}}{2I} = \frac{L_i^2}{2I} = \frac{m_1^2 l^2 v_i^2}{2l^2\left(2m_2 + m_1\right)} = \frac{1}{2}m_1 v_i^2\left(\frac{m_1}{2m_2 + m_1}\right) = K_i\left(\frac{m_1}{2m_2 + m_1}\right)$$

Note that since angular momentum is conserved, we can use either L_i or L_f in the kinetic energy equation; using L_i is somewhat simpler algebraically. The ratio of final to initial kinetic energies is thus

$$\frac{K_f}{K_i} = \frac{m_1}{2m_2 + m_1}$$

(c) What happens once the system starts rotating? Even without the initial kinetic energy of the smaller mass, since all forces present after the collision are conservative the whole system would have enough energy to rotate through 180°, since that would put all of the masses back at the same height. The gravitational potential energy of the system right after the collision is the same as that after rotating through 180°, so the system must rotate at least that much.

After rotating through 180°, the total mechanical energy of the system is unchanged from the point right after the collision. The system will continue rotating through a further maximum angle θ at which point the gain in potential energy equals the kinetic energy right after the collision, K_f. As the system rotates, one of the m_2 masses will go up by an amount $h = l\sin\theta$, and the other m_2 mass will go down by the same amount. The only change in potential energy comes from the smaller m_1 mass moving up by h! We can balance mechanical energy between configurations right after the collision, after rotating through 180°, and after rotating through an additional angle θ. Let the initial horizontal axis of the dumbbell be the zero of potential energy.

$$
\begin{aligned}
\text{after collision:} \quad & K + U = K_f \\
\text{after rotating through 180°:} \quad & K + U = K_f \\
\text{after an additional rotation by } \theta\text{:} \quad & K + U = m_2 gl\sin\theta + m_1 gl\sin\theta - m_2 gl\sin\theta = m_1 gl\sin\theta \\
\text{conservation of mechanical energy} \implies & m_1 gl\sin\theta = K_f = \frac{m_1^2 v_i^2}{2\left(2m_2 + m_1\right)} \\
& \sin\theta = \frac{m_1 v_i^2}{2gl\left(2m_2 + m_1\right)} \\
& \theta = \sin^{-1}\left[\frac{m_1 v_i^2}{2gl\left(2m_2 + m_1\right)}\right]
\end{aligned}
$$

The total angle of rotation is thus $180° + \theta$.

4. Archimedes supposedly was asked to determine whether a crown made for the king consisted of pure gold. Legend has it that he solved this problem by weighing the crown first in air and then in water. Suppose the scale read $7.84\,\mathrm{N}$ in air and $6.84\,\mathrm{N}$ while submersed in water. What should Archimedes have told the king? (Note: $\rho_{\text{water}} = 1000\,\mathrm{kg/m^3}$, $\rho_{\text{gold}} = 19.3 \times 10^3\,\mathrm{kg/m^3}$

Solution: Weighed out of the water, the scale should read the gravitational force of the crown W, meaning its density times volume times g.

$$W = \rho_s V g = 7.84\,\mathrm{N} \tag{4}$$

Weighed in the water, the scale will read the object's weight minus the additional buoyant force B acting upwards, which is itself equal to the object's volume times the density of the fluid times g - the weight of the displaced water.

$$W - B = \rho_s V g - \rho_l V g = V g (\rho_s - \rho_l) = 6.84\,\mathrm{N} \tag{5}$$

We can divide the two equations and solve for the apparent density of the crown ρ_s to see if it matches the known density of gold.

$$\frac{\rho_s - \rho_l}{\rho_s} = \frac{6.84}{7.84} \tag{6}$$

$$\frac{6.84}{7.84}\rho_s = \rho_s - \rho_l \tag{7}$$

$$\rho_s = \frac{\rho_l}{1 - \frac{6.84}{7.84}} = 7840\,\mathrm{kg/m^3} \tag{8}$$

The apparent density of the crown is far less than the expected value for gold, so the crown cannot be made of pure gold. The apparent density is also far less than that of most metals known to the ancient Greeks, like copper, iron, silver, lead, bronze, and brass. That means the crown can't be a mix of any of these metals with gold, since whatever one mixes with gold would have to have a smaller density than $7840\,\mathrm{kg/m^3}$ to come up with this value of ρ_s overall. (Aluminum is far less dense (about $2700\,\mathrm{kg/m^3}$) but was not discovered until 1825.) Of the metals known at the time, only tin at $7260\,\mathrm{kg/m^3}$ seems to have a sufficiently small density to explain the result. It is plausible then that the crown is mostly tin with a thin coating of gold on the outside.

5. Cylindrical pressure vessels are often reinforced by braided fibers, wound at a specific angle. For a thin-walled pipe, two times of forces are crucial to the overall strength of the pipe: the forces trying to expand the pipe in the radial direction, and those trying to elongate the pipe (in engineering parlance, we are talking about hoop stresses and axial stresses[ii]). **(a)** What is the relationship between the radial and axial forces, per unit area? **(b)** The winding reinforces the pipe most effectively when the axial and radial components of the tension in the fiber have the same ratio as the axial and radial forces. Given that constraint, what is the optimal winding angle α?

6. The density of water is $1000\,\mathrm{kg/m^3}$, while that of ice is $916.7\,\mathrm{kg/m^3}$. If a block of ice is placed in water,

---

[ii]Stress is just a generalization of pressure for solid objects, it is a force per unit area just like pressure. You may find the following useful: `http://en.wikipedia.org/wiki/Cylinder_stress`.

what volume fraction of the ice is below the surface?

Solution: Let the block of ice under consideration have an area A exposed to the water, with total thickness in the vertical direction be d, with a thickness x below the water. We will assume that the ice is uniform in the lateral directions, having the same thickness everywhere. There are two forces acting on the block of ice: its weight pulling own, and the buoyant force of the displaced water pushing it upward. The former is easy to find given the density of ice ρ_i and the volume of the block Ad:

$$W = mg = \rho_i A d g \tag{9}$$

The buoyant force is due to the water displaced, and it is only the ice below the surface of volume Ax that displaces any water. The buoyant force is the weight of the displaced water, and given a density ρ_w of water:

$$B = \rho_i A x g \tag{10}$$

In equilibrium, the two forces must be equal:

$$W = \rho_i A d g = B = \rho_i A x g \tag{11}$$

$$\implies \quad \frac{x}{d} = \text{fraction below water} = \frac{\rho_i}{\rho_w} \approx 0.917 \tag{12}$$

Thus, the old adage that nine tenths of an iceberg is underwater is basically correct.

7. Superman attempts to drink water through a very long straw. With his great strength, he achieves maximum possible suction. The walls of the straw do not collapse. Find the maximum height through which he can lift the water.

Solution: Let's say Superman reduces the pressure inside the straw to some value P_o compared to the outside atmosphere of pressure P_a. The net force on a column of water of cross-sectional area A would then be $(P_a - P_o) A$. If this force is to lift the column of liquid through a height h, it must be equal to the weight of a column of said liquid of area A and height h.

$$\text{net suction force} = (P_a - P_o) A \tag{13}$$

$$\text{weight of water pulled up} = \rho h A g \tag{14}$$

What is the best Superman could do? He could reduce the pressure in the straw to zero. There is no lower pressure! In this case, $P_o = 0$, equating the two forces gives:

$$P_a A = \rho h A g \tag{15}$$

$$h = \frac{P}{\rho g} \approx 10.3 \, \text{m} \tag{16}$$

The best he can do is to cause the atmospheric pressure P_a to push up on the water with all its might, but no more. The power of suction is fundamentally limited by the surrounding pressure, Superman or otherwise.

8. Viscosity of most liquids can be represented by an extra "drag" force on a body moving in a liquid, which

is reasonably well approximated by $F_{\text{drag}} = 6\pi\eta R v$, where v is the velocity of the body, η is the viscosity parameter of the fluid, and R is a characteristic dimension of the falling object (the radius, in the case of a sphere). The presence of viscosity leads to a "terminal velocity" of a body falling in a liquid.

Consider a sphere of radius R and density ρ_s falling through a liquid of density ρ_l and viscosity parameter η. Including this new drag force, the buoyant force, and the weight of the object, find an expression for the terminal velocity of the sphere.

Solution: Terminal velocity is when the the object in question reaches a constant maximum velocity, which must be when the net force on the object is zero. Physically, the object's speed has become so high, and the corresponding drag force so great that it manages to balance the object's weight and any other forces. The weight of a sphere of radius R and density ρ_s is

$$F_w = \frac{4}{3}\pi R^3 \rho_s g \tag{17}$$

If we are in a surrounding fluid of density ρ, we must also account for the buoyant force, equal to the weight of the displaced fluid. This is the same as the expression above if we substitute $\rho_s \to \rho$

$$B = \frac{4}{3}\pi R^3 \rho g \tag{18}$$

The drag and buoyant forces will act in one direction, the weight of the object opposing them. A force balance yields, at terminal velocity,

$$0 = F_d + B - F_w \tag{19}$$

$$0 = 6\pi\eta R v + \frac{4}{3}\pi R^3 \rho g - \frac{4}{3}\pi R^3 \rho_s g \tag{20}$$

$$6\pi\eta R v = \frac{4}{3}\pi R^3 \left(\rho_s - \rho\right) g \tag{21}$$

$$v = \frac{2g}{9\eta}R^2 \left(\rho_s - \rho\right) \tag{22}$$

The fact that terminal velocity depends on particle size has many interesting technological applications. (As a quick for-instance: `http://en.wikipedia.org/wiki/Fluidized_bed_reactor`.)