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Quiz 7 Solution

1. A particle of mass $m = 2\,\text{kg}$ experiences a spatially varying potential energy $U(x) = \frac{-2}{x} + \frac{1}{x^2}$, where x is in meters and U is in Joules. In other words, $U(x)$ is the sum total potential energy of the particle (from all sources) for any position x. What is the *stable* equilibrium position of the particle? Recall that $F = -\frac{dU}{dx}$.

Equilibrium points occur when the potential has no spatial variation, or the net force is zero:

$$\frac{dU}{dx} = -F = 0$$

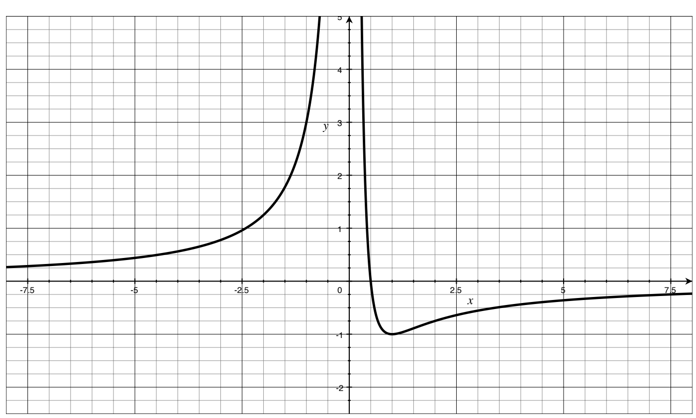
A *stable* equilibrium is characterized by a potential energy that is locally concave upward, or a force that points back toward the original position for any infinitesimal displacement:

$$\frac{d^2U}{dx^2} = -\frac{dF}{dx} > 0$$

A negative spatial derivative of the force means the force for any small displacement tends to push the particle back to where it started, and the positive second derivative of the potential is just applying the second derivative test to prove that we have a local *minimum* in the potential. Think of $U(x)$ as representing a physical landscape - a particle will want to fall into local minima, global minima if possible. With the function given:

$$\begin{aligned} & \frac{dU}{dx} = -F = \frac{2}{x^2} - \frac{2}{x^3} = 0 \\ \Longrightarrow \quad & x_{\text{eq}} = 1 \end{aligned}$$

The only position on the $U(x)$ curve with zero slope is thus at $x = 1\,\text{m}$, this is the only possible equilibrium point. A quick plot confirms this – in the figure below, U is on the y axis, and x is just what you think it is.



Clearly from the plot, the equilibrium is stable. Checking the second derivative:

$$\left.\frac{d^2U}{dx^2}\right|_{x_{\text{eq}}} = -\left.\frac{dF}{dx}\right|_{x_{\text{eq}}} = \left.\left[\frac{-2}{x^3} + \frac{6}{x^4}\right]\right|_{x_{\text{eq}}} = 2 > 0$$

Indeed, we have a local minimum, and therefore a stable equilibrium at $x = 1\,\text{m}$.

Incidentally, why do we not worry about $x = 0$, where the function is not defined? It takes an infinite amount of energy, or an infinite force, to reach $x = 0$. That's just silly.

2. An object of mass m is attached to a spring hung vertically from the ceiling. After attaching the mass m, the spring stretches a distance d from its former equilibrium. If $m = 0.55\,\text{kg}$ and $d = 2\,\text{cm}$, what is the force constant of the spring?

There are two ways to solve this one: force-based, and energy-based. First, the force-based approach.

The mass is at rest, and thus $a = 0$. This means the net force is also zero. The net force is made up of the spring's restoring force and the object's weight, pointing in opposite directions. Pick "up" to be positive y. The spring's displacement points toward $-y$, so the spring force is $-k\Delta x = -k\,(-d) = kd$. The object's weight is just $-mg$.

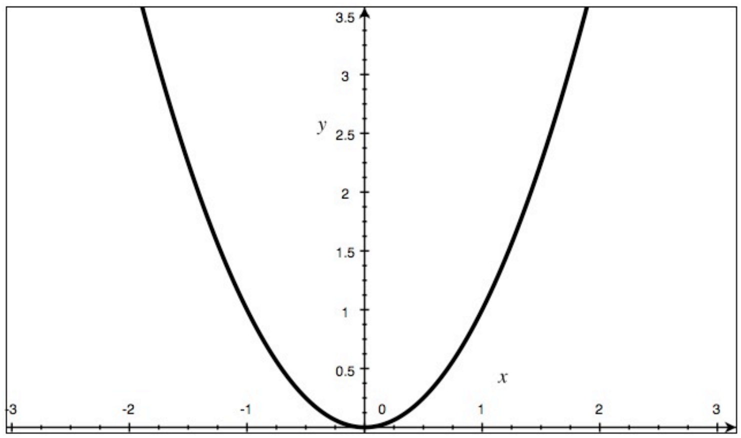
$$\begin{aligned} \vec{\mathbf{F}}_{\text{net}} &= \vec{\mathbf{F}}_{\text{spring}} + \vec{\mathbf{F}}_{\text{gravity}} = kd - mg = 0 \\ \Longrightarrow \quad k &= \frac{mg}{d} \approx 270\,\text{N/m} \end{aligned}$$

How about the energy approach? Let the object's final position define zero gravitational potential energy. We start then with $U_g = mgh$ worth of potential energy. We end with a spring expanded by an amount d, which must have $U_s = \frac{1}{2}kd^2$. Done, right? Wrong. How did the mass move downward? Some external agent must have done work against the spring to push the mass down! The initial mechanical energy *minus* the work done against the spring force must give the final mechanical energy. Note that the spring force is $+ky$, since it acts in the $+y$ direction, while the displacement is in the $-y$ direction. Calling the final position $y = 0$ (consistent with our choice of zero gravitational potential), the mass moves from d to 0.

$$\begin{aligned} K_i + U_i - W_{\text{ext}} &= K_f + U_f \\ mgd - \int_0^d F_{\text{spring}}\,(-dy) &= \frac{1}{2}kd^2 \\ mgd - \int_d^0 (ky)\,(-dy) &= \frac{1}{2}kd^2 \\ mgd - \frac{1}{2}kd^2 &= \frac{1}{2}kd^2 \\ \Longrightarrow \quad k &= \frac{mg}{d} \approx 270\,\text{N/m} \end{aligned}$$

Energy-based approaches are incredibly powerful, but also very sneaky: you have to be very careful about where energy is going, and what work is being done.

3. Consider the potential energy versus position diagram $U(x)$ below. An particle of mass $0.1\,\mathrm{kg}$ subjected to this potential energy function has a *total* energy of $1\,\mathrm{J}$. What is (roughly) its maximum possible displacement? You may assume perfect conservation of potential + kinetic energy.



It is easy to over-think this one. If the particle's maximum energy is $1\,\mathrm{J}$, then the maximum potential energy U it can have is $1\,\mathrm{J}$. When U is maximum, if mechanical energy is conserved it follows that kinetic energy is zero, $K = 0$. If $K = 0$, the particle has reached a turning point, or an extreme point in its motion. Looking at the graph, the particle has a potential energy of $1\,\mathrm{J}$ at approximately $x = +1, -1\,\mathrm{m}$. This means the particle cannot go farther left than $-1\,\mathrm{m}$, or farther right than $+1\,\mathrm{m}$. The displacement is thus $2\,\mathrm{m}$.

Note that you were asked only to consider the potential energy given by the graph – this is the object's *only* potential energy. Gravitational potential is not relevant, and thus neither is conservation of kinetic energy plus gravitational energy.