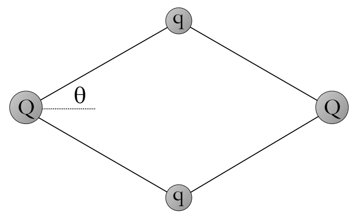
Name & CWID

PH126 Exam I

Instructions

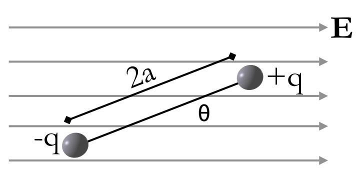
1. **Solve two of the four problems below.** All problems have equal weight.
2. Clearly mark your which problems you have chosen using the tick box.
3. You are allowed 1 sheet of standard 8.5x11 in paper and a calculator.



□ **1.** Four positively charged bodies, two with charge Q and two with charge q, are connected by four unstretchable strings of equal length. In the absence of external forces they assume the equilibrium configuration shown in the diagram.

Show that $\tan^3\theta = q^2/Q^2$**.**

Note: This can be done in two ways. You could show that this relation must hold if the total force on each body, the vector sum of string tension and electrical repulsion, is zero. Or you could write out the expression for the energy U of the assembly and minimize it.



□ **2.** An electric dipole in a uniform electric field E is displaced slightly from its equilibrium position, as shown above. The angle between the dipole axis and the electric field is θ (you may assume θ is small). The separation of the charges is $2a$, and the moment of inertia of the dipole is I.

Assuming the dipole is released from this position, show that its angular orientation exhibits simple harmonic motion with a frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{2qaE}{I}}$$

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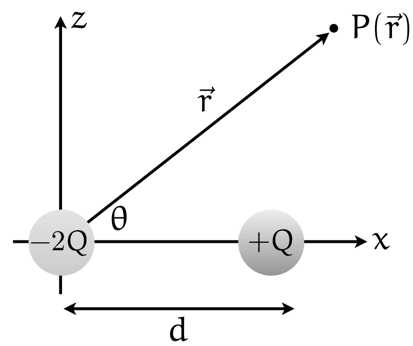

□ 3. The charge distribution shown above is not quite a dipole, but may be considered to be the superposition of a dipole and a monopole.

(a) Find an approximate form for the potential at a point $P(\vec{r})$ far from the charges ($d \ll x, z$) in terms of the radial distance r and angle θ. You may treat the problem in two dimensions if you wish.
(b) Find an approximate form for the electric field at P.

Note: you may find the following approximation useful: $(1+x)^n \approx 1+nx$. See the last exam sheet for formulas relating to spherical coordinates …

□ 4. A sphere of radius R carries a charge density $\rho(r)=cr$, where c is a constant.

(a) Find the total charge Q contained in the sphere.
(b) Find the electric field everywhere.
(c) Find the energy of the configuration.

Note: there are two straightforward ways for the last part: from the energy in the electric field *everywhere*, and from the potential over the charge distribution.

Constants:

$$\begin{aligned} k_e &\equiv 1/4\pi\epsilon_o = 8.98755\times 10^{9}\,\mathrm{N\cdot m^2\cdot C^{-2}} \\ \epsilon_o &= 8.85\times 10^{-12}\,\mathrm{C^2/N\cdot m^2} \\ e &= 1.60218\times 10^{-19}\,\mathrm{C} \\ m_{e^-} &= 9.10938\times 10^{-31}\,\mathrm{kg} \\ m_{p^+} &= 1.67262\times 10^{-27}\,\mathrm{kg} \end{aligned}$$

Basic Equations / Mechanics:

$$0 = ax^2 + bx^2 + c \Longrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\vec{F}_{\text{centr}} = -\frac{mv^2}{r}\,\hat{r} \quad \text{Centripetal}$$

$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad \omega = 2\pi f \quad \text{simple harmonic motion}$$

$$\sum \vec{F} = m\vec{a}$$

$$\sum |\vec{\tau}| = I\alpha = I\,\frac{d^2\theta}{dt^2}$$

Electric Force & Field (static case):

$$\vec{F}_{12} = k_e\,\frac{q_1 q_2}{r_{12}^2}\,\hat{r}_{12} = q_2\vec{E}_1 \qquad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$

$$\vec{E}_1 = \vec{F}_{12}/q_2 = k_e\,\frac{q_1}{r_{12}^2}\,\hat{r}_{12}$$

$$\vec{E} = k_e \sum_i \frac{q_i}{r_i^2}\,\hat{r}_i \to k_e \int \frac{dq}{r^2}\,\hat{r} = k_e \int_V \frac{\rho\hat{r}}{r^2}\,d\tau$$

$$\rho\,d\tau \to \sigma\,da \to \lambda\,dl$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_o \qquad \vec{E} = -\vec{\nabla}V$$

$$\int_V \vec{\nabla}\cdot\vec{E}\,d\tau = \oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_o}\int_V \rho\,d\tau = \frac{q_{\text{encl}}}{\epsilon_o}$$

$$\left(\vec{E}_{\text{above}} - \vec{E}_{\text{below}}\right)\cdot\hat{n} = 4\pi k_e\sigma = \sigma/\epsilon_o \quad \text{sheet of charge with } \sigma$$

$$F_{\text{sheet}} = \frac{\sigma}{2}\left(\vec{E}_{\text{above}} + \vec{E}_{\text{below}}\right)$$

Electric Potential (static case):

$$\Delta V = V_B - V_A = \frac{\Delta U}{q} = -\int_A^B \vec{E}\cdot d\vec{l}$$

$$V_{\text{point}} = k_e\,\frac{q}{r} \to V_{\text{continuous}} = k_e\int\frac{dq}{r} = k_e\int\frac{\rho}{r}\,d\tau$$

$$U_{\text{pair of point charges}} = k_e\,\frac{q_1 q_2}{r_{12}} = V_1 q_2 = V_2 q_1$$

$$U_{\text{system}} = \text{sum over unique pairs} = \sum_{\text{pairs } ij}\frac{k_e q_i q_j}{r_{ij}} \to \frac{1}{2}\int \rho V\,d\tau$$

$$U_{\text{field}} = \frac{\epsilon_o}{2}\int E^2\,d\tau = \frac{1}{2}\int \rho V\,d\tau$$

$$\frac{\partial V}{\partial n} = \vec{\nabla}V\cdot\hat{n} = -\vec{E}\cdot\hat{n} \qquad \text{boundary condition, sheet of chg}$$

$$\frac{\partial V}{\partial n} = -\sigma/\epsilon_o \qquad \text{boundary condition, cond.}$$

Vectors:

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2} \quad \text{magnitude} \qquad \theta = \tan^{-1}\left[\frac{F_y}{F_x}\right] \quad \text{direction}$$

$$\hat{r} = \vec{r}/|\vec{r}| \quad \text{construct any unit vector}$$

let $\vec{a} = a_x\,\hat{x} + a_y\,\hat{y} + a_z\,\hat{z}$ and $\vec{b} = b_x\,\hat{x} + b_y\,\hat{y} + b_z\,\hat{z}$

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z = \sum_{i=1}^{n} a_i b_i = |\vec{a}||\vec{b}|\cos\theta$$

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\,\hat{x} + (a_z b_x - a_x b_z)\,\hat{y} + (a_x b_y - a_y b_x)\,\hat{z}$$

Vector Calculus:

$$d\vec{l} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz \quad \text{length, cartesian}$$

$$d\vec{l} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\varphi}\,r\sin\theta\,d\varphi \quad \text{length, spherical}$$

$$d\tau = dx\,dy\,dz \quad \text{volume, cartesian}$$

$$d\tau = r^2\sin\theta\,dr\,d\theta\,d\varphi \quad \text{volume, spherical}$$

$$\vec{\nabla} = \hat{x}\,\frac{\partial}{\partial x} + \hat{y}\,\frac{\partial}{\partial y} + \hat{z}\,\frac{\partial}{\partial z} \quad \text{cartesian}$$

$$\vec{\nabla} = \hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\,\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \quad \text{spherical}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \quad \text{cartesian}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi} \quad \text{spherical}$$

cartesian (x, y)	**polar** (r, θ)
$r = \sqrt{x^2 + y^2}$	$x = r\cos\theta$
$\theta = \tan^{-1}(y/x)$	$y = r\sin\theta$

	cartesian (x, y, z)
spherical	$r = \sqrt{x^2 + y^2 + z^2}$ $\varphi = \tan^{-1}\left(\frac{y}{x}\right)$ $\theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right)$
	spherical $(r, \varphi\vert_0^{2\pi}, \theta\vert_0^{\pi})$
cartesian	$x = r\sin\theta\cos\varphi$ $y = r\sin\theta\sin\varphi$ $z = r\cos\theta$

	cartesian $(x, y, z,)$
spherical	$\hat{r} = \frac{1}{r}(x\,\hat{x} + y\,\hat{y} + z\,\hat{z})$ $\hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$ $\hat{\theta} = \cos\theta\cos\varphi\,\hat{x} + \cos\theta\sin\varphi\,\hat{y} - \sin\theta\,\hat{z}$ $\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$
	spherical $(r, \varphi\vert_0^{2\pi}, \theta\vert_0^{\pi})$
cartesian	$\hat{x} = \sin\theta\cos\varphi\,\hat{r} + \cos\theta\cos\varphi\,\hat{\theta} - \sin\varphi\,\hat{\varphi}$ $\hat{y} = \sin\theta\sin\varphi\,\hat{r} + \cos\theta\sin\varphi\,\hat{\theta} + \cos\varphi\,\hat{\varphi}$ $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

Calculus of possible utility:

$$\int \frac{1}{x}\,dx = \ln x + c$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$$

$$\int \frac{x}{a^2+x^2}\,dx = \frac{1}{2}\ln|a^2 + x^2| + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$$

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 + x^2}} + C$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\frac{d}{dx}\tan x = \sec^2 x = \frac{1}{\cos^2 x}$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\frac{1}{u} = \frac{-1}{u^2}\frac{du}{dx}$$