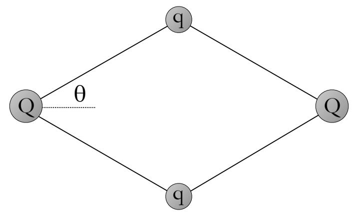
PH126 Exam I Solutions



□ 1. Four positively charged bodies, two with charge Q and two with charge q, are connected by four unstretchable strings of equal length. In the absence of external forces they assume the equilibrium configuration shown in the diagram.

Show that $\tan^3\theta = q^2/Q^2$**.**

Note: This can be done in two ways. You could show that this relation must hold if the total force on each body, the vector sum of string tension and electrical repulsion, is zero. Or you could write out the expression for the energy U of the assembly and minimize it.

Energy-based approach. Let the length of the strings connecting adjacent Q and q charges be d. Call the distance between the two Q charges horizontally l, and the vertical distance between the two q charges h. Using trigonometry, then:

$$\cos\theta = \frac{l/2}{d} = \frac{l}{2d}$$
$$\sin\theta = \frac{h/2}{d} = \frac{h}{2d}$$

The total potential energy of this system can be found by adding the potential energies of all unique pairs of charges, recalling that for a pair of point charges q_1 and q_2 separated by a distance r_{12} the potential energy is $k_e q_1 q_2/r_{12}$. We also note that there are four equivalent pairings of the q and Q charges, all separated by a distance d.

$$\begin{aligned} U &= \frac{k_e Q^2}{l} + \frac{k_e q^2}{h} + 4\frac{k_e Q q}{d} \\ &= \frac{k_e Q^2}{2d\cos\theta} + \frac{k_e q^2}{2d\sin\theta} + 4\frac{k_e Q q}{d} \end{aligned}$$

Now we have the potential energy of the entire system as a function of the angle θ. Notice that the last term - the potential energy due to the charges fixed by the string - does not depend on θ, since the distance between adjacent q and Q charges is fixed by the length of the string.

At equilibrium, this potential energy should be at a *minimum* with respect to any angular variation. If $U(\theta)$ should be at a minimum, we must have $dU/d\theta = 0$:

$$\frac{dU}{d\theta} = \frac{d}{d\theta}\left[\frac{k_e Q^2}{2d\cos\theta} + \frac{k_e q^2}{2d\sin\theta} + 4\frac{k_e Q q}{d}\right] = 0$$
$$\frac{dU}{d\theta} = \frac{k_e Q^2}{d}\frac{\sin\theta}{2\cos^2\theta} - \frac{k_e q^2}{d}\frac{\cos\theta}{2\sin^2\theta} = 0$$

Solving the equation above,

$$\frac{k_e Q^2}{d}\frac{\sin\theta}{2\cos^2\theta} = \frac{k_e q^2}{d}\frac{\cos\theta}{2\sin^2\theta}$$
$$Q^2\frac{\sin\theta}{2\cos^2\theta} = q^2\frac{\cos\theta}{2\sin^2\theta}$$
$$\frac{q^2}{Q^2} = \frac{\sin^3\theta}{\cos^3\theta} = \tan^3\theta$$

Now, we have forgotten to be careful about one thing: is this a maximum, a minimum, or an inflection point? Setting $dU/d\theta = 0$ only ensures we have found one of the three; recall from Calculus I *which one it is* depends on the sign of $d^2U/d\theta^2$. One can argue on physical grounds that it must be a minimum, but mathematically one must show that $d^2U/d\theta^2 > 0$ to be certain.

Finding the second derivative of $U(\theta)$ is rather messy; you should find something like this once you grind through it:

$$\frac{d^2U}{d\theta^2} = \frac{d}{d\theta}\frac{dU}{d\theta} = \frac{k_e}{2d}\left(\frac{Q^2}{\cos\theta} + \frac{q^2}{\sin\theta}\right)$$

For the present problem, the angle θ can only be between 0 and 90° without breaking the strings. The equation above is positive over that entire range of angles (though singular at the endpoints 0 and 90°), which means that $d^2U/d\theta^2 > 0$ for any physically possible choice of θ, and we have indeed found a *minimum* of potential energy, rather than a maximum or inflection point. Thus, our condition represents a *stable* situation.

Force-based approach. First, refer to the figure below, where we have drawn a simple free-body diagram about one of the q charges, and one of the Q charges.

We will call the force between adjacent Q and q charges F_{qQ}, the force between two q charges F_{qq}, the force between two Q charges F_{QQ}, and finally, the tension in the strings is T. All four strings must have the same tension, based on the symmetry of the system and Newton's third law. Since we know the distances between the charges (see above), we already know the electrostatic forces involved:

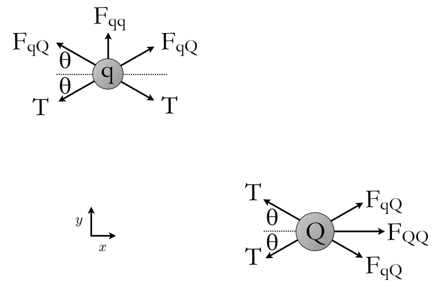


Figure 1: *Problem 3: free-body diagram*

$$\begin{aligned} F_{QQ} &= \frac{k_e Q^2}{4d^2\cos^2\theta} \\ F_{qq} &= \frac{k_e q^2}{4d^2\sin^2\theta} \\ F_{qQ} &= \frac{k_e Q q}{d^2} \end{aligned}$$

Next, focus on one of the q charges. We will pick the uppermost one just to be concrete. As indicated in the free body diagram above, there will be two repulsive F_{qQ} forces from the two adjacent Q charges, and these forces will be directed at an angle θ above the indicated x axis. The string tensions will act opposite these two repulsive forces. At equilibrium, all forces must sum to zero. Summing the forces along the x and y axes, we have:

$$\begin{aligned} \text{on } q \text{ charge:} \quad \sum F_x &= F_{qQ}\cos\theta - F_{qQ}\cos\theta + T\cos\theta - T\cos\theta = 0 \\ \sum F_y &= 2F_{qQ}\sin\theta - 2T\sin\theta + F_{qq} = 0 \end{aligned}$$

The forces in the x direction give us nothing useful, but those in the y direction do. Plugging in our expressions for the forces:

$$\frac{2k_e Q q}{d^2}\sin\theta - 2T\sin\theta + \frac{k_e q^2}{4d^2\sin^2\theta} = 0 \tag{1}$$

This looks useful, but it is not enough. We must eliminate the tension T, and the only way to get enough equations to do so is to also perform a force balance around one of the Q charges. Pick the rightmost one:

$$\begin{aligned} \text{on } Q \text{ charge:} \quad \sum F_x &= F_{QQ} + 2F_{qQ}\cos\theta - 2T\cos\theta \\ \sum F_y &= F_{qQ}\sin\theta - F_{qQ}\sin\theta + T\sin\theta - T\sin\theta = 0 \end{aligned}$$

This time, the y force balance is useless, but the x force balance gives us another interesting equation.

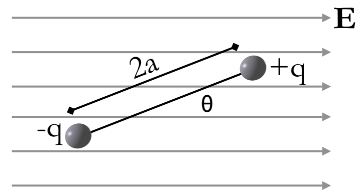
Again, plugging in our expressions for the forces:

$$\frac{k_e Q^2}{4d^2 \cos^2\theta} + \frac{2k_e Q q}{d^2}\cos\theta - 2T\cos\theta = 0 \tag{2}$$

Now: compare equations (1) and (2). We can solve both equations for $2T$, and eliminate the tensions entirely:

$$\begin{aligned}
2T &= \frac{2k_e Q q}{d^2} + \frac{k_e q^2}{4d^2\sin^3\theta} \qquad \text{from (1)} \\
2T &= \frac{2k_e Q q}{d^2} + \frac{k_e Q^2}{4d^2\cos^3\theta} \qquad \text{from (2)} \\
\implies \quad \frac{k_e q^2}{4d^2\sin^3\theta} &= \frac{k_e Q^2}{4d^2\cos^3\theta} \\
\frac{q^2}{Q^2} &= \frac{\sin^3\theta}{\cos^3\theta} = \tan^3\theta
\end{aligned}$$

Thus, as it must, the force-based approach yields the same answer as the energy-based approach.



□ 2. An electric dipole in a uniform electric field E is displaced slightly from its equilibrium position, as shown above. The angle between the dipole axis and the electric field is θ (you may assume θ is small). The separation of the charges is $2a$, and the moment of inertia of the dipole is I.

Assuming the dipole is released from this position, show that its angular orientation exhibits simple harmonic motion with a frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{2qaE}{I}}$$

Define the positive x axis to be in the direction of the electric field, and the positive y axis perpendicular to it in the upward direction. This means the z axis points out of the page for a right-handed coordinate system.

On the $+q$ charge, there will be force $F_+ = qE$ along $\hat{\mathbf{x}}$, and on the $-q$ charge, a force $F_- = -qE$ (along $-\hat{\mathbf{y}}$). Both of these forces will try to cause the dipole to rotate and orient itself along the electric field; that is, both will result in a clockwise torque about the center of the dipole. The sum of these torques must equal the dipole's moment of inertia I times the resulting angular acceleration α. Let the position of the $+q$ charge be defined by a vector $\vec{r}_+$ whose origin is at the dipole center, and similarly $\vec{r}_-$ will give the position of the $-q$ charge. We also define a unit vector $\hat{\mathbf{r}}$ pointing from the $-q$ to the $+q$ charge. Finally, remember that a clockwise rotation defines a *negative* torque - this is the version of the right-hand rule for torques.[i]

$$\begin{aligned}\sum \vec{\tau} &= \left(\vec{r}_+ \times \vec{F}_+\right) + \left(\vec{r}_- \times \vec{F}_-\right) = qEa\,\hat{\mathbf{r}} \times \hat{\mathbf{x}} + (-qE)\,(-\hat{\mathbf{r}} \times \hat{\mathbf{x}}) \\ &= qEa\,(-\hat{\mathbf{z}}\sin\theta) + qEa\,(-\hat{\mathbf{z}}\sin\theta) = -2qaE\sin\theta\,\hat{\mathbf{z}} = I\vec{\alpha} \\ \sum |\vec{\tau}| &= -2qEa\sin\theta = I|\vec{\alpha}|\end{aligned}$$

We could have avoided the vector baggage right off the bat, if we just chose the resultant torque to be negative based on the right-hand rule. In order to show simple harmonic motion, we need to show in this case that $\alpha = -\omega^2\theta$. Recalling the definition of α, we have:

[i]We have to pick the sign because $\vec{\tau}$, resulting from a cross-product, is a pseudovector.

$$I\alpha = I\frac{d^2\theta}{dt^2} = -2qEa\sin\theta$$

If the angle θ is small, we can approximate $\sin\theta$ by the first term in its Taylor expansion (a "first-order" approximation):

$$\text{Taylor expansion} \qquad \sin\theta = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\theta^{2n+1} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

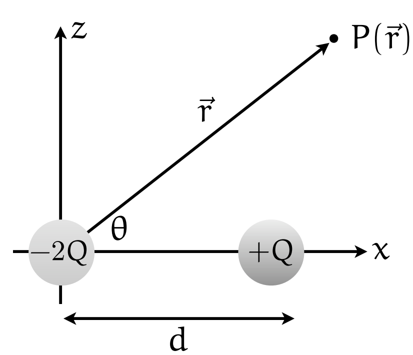
$$\text{small } \theta: \qquad \sin\theta \approx \theta$$

Using this approximation,

$$I\frac{d^2\theta}{dt^2} \approx -2qEa\theta \qquad \text{or} \qquad \frac{d^2\theta}{dt^2} \approx -\frac{2qEa}{I}\theta$$

This is our beloved differential equation for simple harmonic motion, viz., $\frac{d^2\theta}{dt^2} = -\omega^2\theta$, and thus for small θ,

$$\omega \approx \sqrt{\frac{2qaE}{I}} \qquad \text{or} \qquad f \approx \frac{1}{2\pi}\sqrt{\frac{2qaE}{I}}$$



□ 3. The charge distribution shown above is not quite a dipole, but may be considered to be the superposition of a dipole and a monopole.

(a) Find an approximate form for the potential at a point $P(\vec{r})$ far from the charges ($d \ll x, z$) in terms of the radial distance r and angle θ. You may treat the problem in two dimensions if you wish.
(b) Find an approximate form for the electric field at P.

Note: you may find the following approximation useful: $(1+x)^n \approx 1 + nx$. See the last exam sheet for formulas relating to spherical coordinates ...

One thing to recognize right off the bat is that this charge distribution is equivalent to a dipole plus one extra negative charge at the origin:

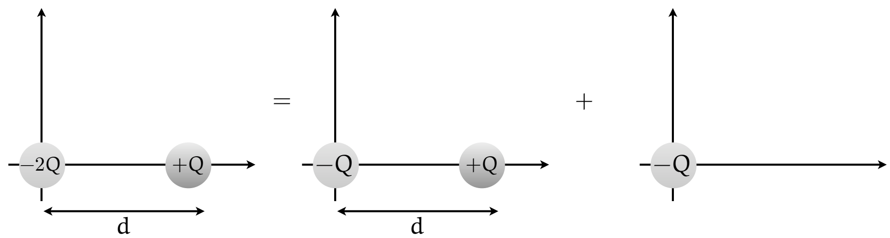


Figure 2: *Our charge distribution is equivalent to a dipole plus a point charge.*

Thus, the solution to our problem is our usual dipole potential plus the potential of a point charge. First, let's consider the dipole alone, and we can add the point charge in later. We can readily write down the potential for the dipole at the point $\mathcal{P}$ – it is just a superposition of the potential due to each of the charges alone. We'll work in two dimensions, so long as we have the option.

$$V_{\text{dipole}}(x,z) = k_e \left[\frac{q}{\sqrt{(x-d)^2+z^2}} + \frac{-q}{\sqrt{x^2+z^2}} \right] \tag{3}$$

Since we are assuming $r \gg d$, we simplify the denominator in the first term a bit, remembering that $r^2 = x^2 + y^2$:

$$\begin{aligned} \frac{1}{\sqrt{(x-d)^2+y^2+z^2}} &= \frac{1}{\sqrt{x^2-2xd+d^2+z^2}} = \frac{1}{\sqrt{x^2+z^2}} \frac{1}{\sqrt{1-\frac{2xd}{\sqrt{x^2+z^2}}+\frac{d^2}{\sqrt{x^2+z^2}}}} \\ &= \frac{1}{r}\frac{1}{1-2xd/r^2+d^2/r^2} \approx \frac{1}{r}\frac{1}{\sqrt{1-2xd/r^2}} \approx \frac{1}{r}\left(1+\frac{xd}{r^2}\right) \end{aligned} \tag{4}$$

Here we used the given (binomial) approximation once again in the very last step. Substituting this in to our expression for the dipole potential above,

$$V_{\text{dipole}}(x,z) \approx \frac{k_e q}{r}\left(1+\frac{xd}{r^2}\right) - \frac{k_e q}{r} = \frac{k_e q}{r}\frac{xd}{r^2} = \frac{k_e q d \cos\theta}{r^2} \tag{5}$$

In the last step, we noted that $x/r = \cos\theta$, using the angle as given in the figure. This is the potential due to the dipole alone; for the full problem, we need only add in the potential due to a charge $-q$ at the origin:

$$V_{\text{tot}}(x,z) \approx V_{\text{dipole}}(x,z) + V_{-q}(x,z) = \frac{k_e q d \cos\theta}{r^2} - \frac{k_e q}{r} = \frac{k_e q}{r}\left(\frac{d\cos\theta}{r} - 1\right) \tag{6}$$

The field is no problem at all, remembering that $\vec{E} = -\vec{\nabla} V$ (and that we are in spherical coordinates). First, the radial part:

$$E_r = -\frac{dV}{dr}\hat{\mathbf{r}} \approx \frac{2k_e q d\cos\theta}{r^3}\hat{\mathbf{r}} - \frac{k_e q}{r^2}\hat{\mathbf{r}} = \frac{k_e q}{r^2}\left(\frac{2d\cos\theta}{r} - 1\right)\hat{\mathbf{r}} \tag{7}$$

Next, the angular part:

$$E_\theta = -\frac{1}{r}\frac{dV}{d\theta}\hat{\theta} \approx \frac{k_e q d\sin\theta}{r^3}\hat{\theta} \tag{8}$$

In total,

$$\vec{E} \approx \frac{k_e q}{r^2}\left(\frac{2d\cos\theta\,\hat{\mathbf{r}}}{r} - \hat{\mathbf{r}} + \frac{d\sin\theta\,\hat{\theta}}{r}\right) = \frac{k_e q d}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}\right) - \frac{k_e q}{r^2}\hat{\mathbf{r}} \tag{9}$$

Just like the potential, the field is the superposition of a dipole and a single point charge $-q$.

□ **4.** A sphere of radius R carries a charge density $\rho(r) = cr$, where c is a constant.

(a) Find the total charge Q contained in the sphere.
(b) Find the electric field everywhere.
(c) Find the energy of the configuration.

Note: there are two straightforward ways for the last part: from the energy in the electric field *everywhere*, and from the potential over the charge distribution.

The total charge is found by integrating the charge density through the volume of the sphere. Remembering to use the differential volume element in spherical coordinates,

$$\begin{aligned} Q_{\text{tot}} &= \int_0^\pi d\theta \int_0^{2\pi} d\varphi \int_0^R \rho(r)\, r^2 \sin\theta\, dr\, d\theta\, d\varphi = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \int_0^R cr\, r^2 \sin\theta\, dr\, d\theta\, d\varphi \\ &= 4\pi c \int_0^R r^3\, dr = \left[4\pi c \frac{r^4}{4}\right]\Bigg|_0^R = \pi c R^4 \end{aligned} \tag{10}$$

The charge distribution is spherically symmetric (ρ does not depend on θ or φ), so for $r > R$ the field looks like that of a point charge of magnitude Q_{tot}:

$$\vec{E} = \frac{kQ_{\text{tot}}}{r^2}\hat{\mathbf{r}} = \frac{k\pi c R^4}{r^2}\hat{\mathbf{r}} = \frac{cR^4}{4\epsilon_o r^2} \qquad (r > R) \tag{11}$$

For points inside the sphere, $r < R$, we need only worry about the charge contained within a sphere of radius r, which can be found from the integral above if we replace the upper limit with r instead of R.

$$\vec{E} = \frac{kQ(r)}{r^2}\hat{\mathbf{r}} = \frac{k\pi c r^4}{r^2}\hat{\mathbf{r}} = k\pi c r^2\,\hat{\mathbf{r}} = \frac{cr^2}{4\epsilon_o}\,\hat{\mathbf{r}} \qquad (r \leqslant R) \tag{12}$$

Once we have the electric field everywhere, the easiest way to find the energy is to integrate the square of the electric field everywhere. We'll have to break this up into two integrals: one for radii less than R and one for radii greater than R, since the field is different in these two regions.

$$
\begin{aligned}
U_{\text{field}} &= \frac{\epsilon_o}{2} \int E^2 \, d\tau &(13)\\
&= \frac{\epsilon_o}{2} \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \int_0^{R} \left(\frac{cr^2}{4\epsilon_o} \hat{\mathbf{r}} \right) \cdot \left(\frac{cr^2}{4\epsilon_o} \hat{\mathbf{r}} \right) r^2 \sin\theta \, dr \, d\theta \, d\varphi &(14)\\
&\quad + \frac{\epsilon_o}{2} \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \int_R^{\infty} \left(\frac{cR^4}{4\epsilon_o r^2} \hat{\mathbf{r}} \right) \cdot \left(\frac{cR^4}{4\epsilon_o r^2} \hat{\mathbf{r}} \right) r^2 \sin\theta \, dr \, d\theta \, d\varphi &(15)\\
&= \frac{\epsilon_o}{2} \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \int_0^{R} \frac{c^2 r^6}{16\epsilon_o^2} \, dr + \frac{\epsilon_o}{2} \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \int_R^{\infty} \frac{c^2 R^8}{16\epsilon_o^2 r^2} \, dr &(16)\\
&= \frac{c^2\pi}{8\epsilon_o} \left[\frac{r^7}{7} \right] \Bigg|_0^R + \frac{c^2\pi R^8}{8\epsilon_o} \left[\frac{-1}{r} \right] \Bigg|_R^{\infty} &(17)\\
&= \frac{c^2\pi}{8\epsilon_o} \left(\frac{R^7}{7} + R^7 \right) = \frac{c^2\pi R^7}{7\epsilon_0} &(18)
\end{aligned}
$$

We could also find the energy of the system by integrating the potential times charge density through the volume of the sphere:

$$
U_{\text{field}} = \frac{1}{2} \int \rho V \, d\tau \tag{19}
$$

Since the integrand is non-zero only in the region where we have charge density – i.e., for $r < R$ – we only need the potential over that region as well. We can get the potential V from $\vec{E}$ readily by integration. In order to find the potential at a distance r from the center of the sphere (still with $r < R$), we'll need to integrate $\vec{E} \cdot d\vec{l}$ from infinity down to r, as if we are brining in the charge to build up the sphere bit by bit. Since $\vec{E}$ is conservative, we can integrate over any path we like, so we may as well make it a nice radial path, $\hat{\mathbf{r}}\, dr$. As with our previous calculation, we'll have to break the integral up into two regions, one outside the sphere, and one within the sphere, since the fields are different in those two regions.

$$
\begin{aligned}
V(r) &= -\int_{\infty}^{r} \vec{E} \cdot d\vec{l} = -\int_{\infty}^{r} \vec{E} \cdot \hat{\mathbf{r}} \, dr = \int_{\infty}^{R} \frac{cR^4}{4\epsilon_o r^2} \, dr - \int_R^{r} \frac{cr^2}{4\epsilon_o} \, dr \\
&= \frac{cR^4}{4\epsilon_o} \left[\frac{1}{r} \right] \Bigg|_{\infty}^{R} - \frac{c}{4\epsilon_o} \left[\frac{r^3}{3} \right] \Bigg|_R^{r} = \frac{cR^3}{3\epsilon_o} - \frac{cr^3}{12\epsilon_o} + \frac{cR^3}{12\epsilon_o} = \frac{4cR^3}{12\epsilon_o} - \frac{cr^3}{12\epsilon_o} = \frac{c}{3\epsilon_o} \left(R^3 - \frac{r^3}{4} \right) \qquad (20)
\end{aligned}
$$

Once we have the potential as a function of r, we can integrate ρV through the volume of the sphere to find the energy:

$$U_{\text{field}} = \frac{1}{2}\int \rho V\, d\tau = \frac{1}{2}\int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \int_0^{R} cr\left(\frac{c}{3\epsilon_o}\right)\left(R^3 - \frac{r^3}{4}\right) r^2 \sin\theta \, dr\, d\theta\, d\varphi \tag{21}$$

$$= \frac{4\pi c^2}{6\epsilon_o}\left[\frac{R^3 r^4}{4} - \frac{r^7}{28}\right]\Bigg|_0^R = \frac{2\pi c^2}{3\epsilon_o}\left[\frac{R^7}{4} - \frac{R^7}{28}\right] = \frac{\pi c^2 R^7}{7\epsilon_o} \tag{22}$$

As it must be, the potential and field methods yield the same result. In my opinion, the field method is somewhat easier in this case, particularly since you were already asked to find the field in the previous part. Still, there is always more than one way to do a problem.