Name & CWID

PH126 Exam III

Instructions

1. **Solve two of the five problems below.**
2. The bonus question is optional, and worth a maximum of +50% of one question.
3. Clearly mark your which problems you have chosen using the tick box.
4. Do your work on separate sheets. Staple them to this exam paper when you are finished.
5. You are allowed 1 sheet of standard 8.5x11 in paper and a calculator.

□ **1.** A particle of mass m is subject to a constant force F along the x axis. If it starts from rest at the origin at time $t=0$, find its position x as a function of time, using relativistic dynamics. Recall that Newton's second law in relativistic form is

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \text{with} \qquad \vec{p} \equiv \frac{m\vec{v}}{\sqrt{1-v^2/c^2}} \tag{1}$$

Note the following useful integral:

$$\int \frac{x}{\sqrt{1+ax^2}}\, dx = \frac{1}{a}\sqrt{1+ax^2} + C \tag{2}$$

□ **2.** In a perfect conductor, the conductivity is infinite, so $\vec{E}=0$, and any net charge resides on the surface (just as it does for an *imperfect* conductor in electrostatics).

(a) Show that the magnetic field is constant (i.e., $\partial\vec{B}/\partial t=0$) inside a perfect conductor.
(b) Show that the magnetic flux through a perfectly conducting loop is constant.

A *superconductor* has infinite conductivity, but is more than a merely perfect conductor: it has the additional property that the (constant) $\vec{B}$ inside is in fact *zero*. This "flux exclusion" is known as the Meissner effect, perfect diamagnetism.

(c) Show that the there is no volume current density in a superconductor, and therefore any current in a superconductor must be confined to the surface.

□ **3.** Suppose

$$\vec{E}(r,\theta,\varphi,t) = E_\varphi\,\hat{\boldsymbol{\varphi}} = A\frac{\sin\theta}{r}\left[\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right]\hat{\boldsymbol{\varphi}} \qquad \text{with} \qquad \frac{\omega}{k}=c \tag{3}$$

This is, in fact, the simplest possible spherical wave. It may be convenient to define $u\equiv kr-\omega t$ in your calculations.

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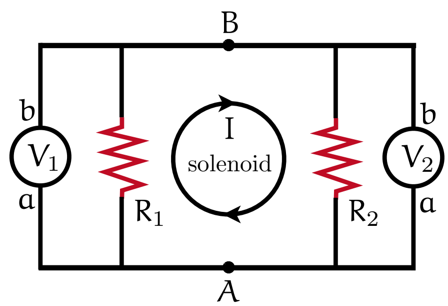
(a) Show that $\vec{E}$ obeys the two Maxwell equations $\nabla\cdot\vec{E}=\rho/\epsilon_o$ and $\nabla\times\vec{E}=-\partial\vec{B}/\partial t$ in vacuum.
(b) Find the associated magnetic field $\vec{B}$ up to an overall constant.
(c) Show that the third Maxwell equation $\nabla\cdot\vec{B}=0$ is satisfied.

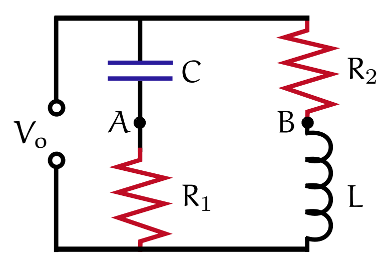
□ **4.** The current in a long solenoid is increasing linearly with time, so that the flux is proportional to t: $\Phi_B=\alpha t$. Two voltmeters are connected to the diametrically opposite points (A and B), together with resistors R_1 and R_2, as shown below. Assume the voltmeters are ideal (infinite input resistance, zero current draw), and that a voltmeter registers the quantity

$$\int_a^b \vec{E}\cdot d\vec{l} \tag{4}$$

between the terminals and through the meter. Find the reading on each voltmeter.



□ **5.** Given the circuit below, show that if the condition $R_1R_2=L/C$ is satisfied, the difference in voltage between points A and B will be zero at *any* frequency.



6. Bonus: The electrical conductivity of a certain crystal with respect to orthogonal axes x_1, x_2, x_3 is represented by this second-rank tensor:

$$\sigma_{ij}=\begin{bmatrix}25 & 0 & 0\\ 0 & 7 & -3\sqrt{3}\\ 0 & -3\sqrt{3} & 13\end{bmatrix}\times 10^7\ (\Omega\cdot\mathrm{m})^{-1} \tag{5}$$

An electric field of magnitude 5 V/m is applied to the crystal in the direction corresponding to the unit vector $(0,\frac{1}{2},\frac{\sqrt{3}}{2})$. Find the current density $\vec{j}$ that results. *Bonus is worth* 1/2 *a normal question.*

Constants:

$$
\begin{aligned}
k_e &\equiv 1/4\pi\epsilon_o = 8.98755\times10^{9}\ \mathrm{N\cdot m^2\cdot C^{-2}}\\
\epsilon_o &= 8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}\\
\mu_o &\equiv 4\pi\times10^{-7}\ \mathrm{T\cdot m/A}\\
c^2 &= 1/\mu_o\epsilon_o \approx \left(3\times10^{8}\ \mathrm{m/s}\right)^2\\
e &= 1.60218\times10^{-19}\ \mathrm{C}
\end{aligned}
$$

Maxwell's equations, Lorentz, & continuity
(without polarization or magnetization)

$$\vec{F}_B = q\vec{v}\times\vec{B}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu_o\vec{j} + \epsilon_o\mu_o\frac{\partial\vec{E}}{\partial t}$$

$$\partial\rho/\partial t + \nabla\cdot\vec{j} = 0$$

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0\epsilon_r} = \frac{1}{\epsilon_0}\int_V \rho\, dV$$

$$\oint_S \vec{B}\cdot d\vec{A} = 0$$

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot d\vec{A}$$

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_o\int_S \vec{j}\cdot d\vec{A} + \epsilon_o\mu_o\frac{\partial}{\partial t}\int_S \vec{E}\cdot d\vec{A}$$

$$\int_S \vec{j}\cdot d\vec{A} = -\frac{d}{dt}\int_V \rho\, dV$$

$$\text{Note } c^2 = \frac{1}{\mu_0\epsilon_0}$$

Electric Force & Field (static case):

$$\vec{F}_{12} = k_e\frac{q_1 q_2}{r_{12}^2}\,\hat{r}_{12} = q_2\vec{E}_1 \qquad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$

$$\vec{E}_1 = \vec{F}_{12}/q_2 = k_e\frac{q_1}{r_{12}^2}\,\hat{r}_{12}$$

$$\vec{E} = k_e\sum_i \frac{q_i}{r_i^2}\,\hat{r}_i \to k_e\int\frac{dq}{r^2}\,\hat{r} = k_e\int_V \frac{\rho\hat{r}}{r^2}\,d\tau$$

$$\rho\, d\tau \to \sigma\, da \to \lambda\, dl$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_o \qquad \vec{E} = -\vec{\nabla}V$$

Electric Potential (static case):

$$\Delta V = V_B - V_A = \frac{\Delta U}{q} = -\int_A^B \vec{E}\cdot d\vec{l}$$

$$V_{\text{point}} = k_e\frac{q}{r} \to V_{\text{continuous}} = k_e\int\frac{dq}{r} = k_e\int\frac{\rho}{r}\,d\tau$$

$$U_{\text{pair of point charges}} = k_e\frac{q_1 q_2}{r_{12}} = V_1 q_2 = V_2 q_1$$

$$U_{\text{field}} = \frac{\epsilon_o}{2}\int E^2\, d\tau = \frac{1}{2}\int \rho V\, d\tau$$

Field transformations
(primed frame moves with v relative to unprimed along x)

$$E'_x = E_x \qquad E'_y = \gamma\left(E_y - vB_z\right) \qquad E'_z = \gamma\left(E_z + vB_y\right)$$

$$B'_x = B_x \qquad B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

Current, Resistance, & Impedance:

$$I = \int_S \vec{J}\cdot d\vec{A} \xrightarrow{\text{uniform j}} I = \frac{dQ}{dt} = nqAv_d$$

$$j = \sum_k n_k q_k v_k \xrightarrow{\text{uniform j}} j = \frac{I}{A} = nqv_d$$

$$R = \Delta V/I \quad \text{or} \quad \vec{j} = \sigma\vec{E} \quad \text{Ohm, isotropic}$$

$$j_i = \sum_{j=1}^{3}\sigma_{ij}E_j \quad \text{Ohm, anisotropic/general}$$

$$Z_C = \frac{1}{i\omega C} \qquad Z_L = i\omega C \qquad Z_R = R$$

Vector Calculus:

$$d\vec{l} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\varphi}\,r\sin\theta\,d\varphi \quad \text{cartesian, spherical}$$

$$d\tau = dx\,dy\,dz \quad = r^2\sin\theta\,dr\,d\theta\,d\varphi \quad \text{cartesian, spherical}$$

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \quad \text{cartesian}$$

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \quad \text{spherical}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \quad \text{cartesian}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, F_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi} \quad \text{spherical}$$

$$\vec{\nabla}\times\vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\varphi\right) - \frac{\partial v_\theta}{\partial\varphi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(r v_\varphi\right)\right]\hat{\theta}$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r v_\theta\right) - \frac{\partial v_r}{\partial\theta}\right]\hat{\varphi} \quad \text{spherical}$$

	cartesian (x, y, z)
spherical	$r = \sqrt{x^2+y^2+z^2}$ $\varphi = \tan^{-1}\left(\frac{y}{x}\right)$ $\theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right)$
	spherical $(r, \varphi\vert_0^{2\pi}, \theta\vert_0^{\pi})$
cartesian	$x = r\sin\theta\cos\varphi$ $y = r\sin\theta\sin\varphi$ $z = r\cos\theta$

	cartesian $(x, y, z,)$
spherical	$\hat{r} = \frac{1}{r}\left(x\,\hat{x} + y\,\hat{y} + z\,\hat{z}\right)$ $\hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$ $\hat{\theta} = \cos\theta\cos\varphi\,\hat{x} + \cos\theta\sin\varphi\,\hat{y} - \sin\theta\,\hat{z}$ $\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$
	spherical $(r, \varphi\vert_0^{2\pi}, \theta\vert_0^{\pi})$
cartesian	$\hat{x} = \sin\theta\cos\varphi\,\hat{r} + \cos\theta\cos\varphi\,\hat{\theta} - \sin\varphi\,\hat{\varphi}$ $\hat{y} = \sin\theta\sin\varphi\,\hat{r} + \cos\theta\sin\varphi\,\hat{\theta} + \cos\varphi\,\hat{\varphi}$ $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

Relativity

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Delta t'_{\text{moving}} = \gamma\Delta t_{\text{stationary}} = \gamma\Delta t_p$$

$$L'_{\text{moving}} = \frac{L_{\text{stationary}}}{\gamma} = \frac{L_p}{\gamma}$$

$$x' = \gamma\left(x - vt\right)$$

$$\Delta t' = t'_1 - t'_2 = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)$$

$$p = \gamma m v$$

$$v_{\text{obj}} = \frac{v + v'_{\text{obj}}}{1 + \frac{v v'_{\text{obj}}}{c^2}} \qquad v'_{\text{obj}} = \frac{v_{\text{obj}} - v}{1 - \frac{v v_{\text{obj}}}{c^2}}$$