UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

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Problem Set 1: Solutions

1. *Variant on Griffiths 1.7.* Find the separation vector $\vec{\boldsymbol{\imath}}=\vec{\mathbf{r}}-\vec{\mathbf{r}}'$ from the source point $\vec{\mathbf{r}}'=(3,4,5)$ to the field point $\vec{\mathbf{r}}=(7,2,17)$. Determine its magnitude $|\vec{\boldsymbol{\imath}}|$ and construct the corresponding unit vector $\hat{\boldsymbol{\imath}}$.

$$\begin{aligned}
\vec{\boldsymbol{\imath}} &= \vec{\mathbf{r}}-\vec{\mathbf{r}}' = (7-3)\,\hat{\mathbf{x}} + (2-4)\,\hat{\mathbf{y}} + (17-5)\,\hat{\mathbf{z}} = 4\,\hat{\mathbf{x}} - 2\,\hat{\mathbf{y}} + 12\,\hat{\mathbf{z}} \\
|\vec{\boldsymbol{\imath}}| &= \sqrt{4^2+(-2)^2+12^2} = 2\sqrt{41} \approx 12.8 \\
\hat{\boldsymbol{\imath}} &= \frac{\vec{\mathbf{r}}-\vec{\mathbf{r}}'}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = \frac{4\,\hat{\mathbf{x}} - 2\,\hat{\mathbf{y}} + 12\,\hat{\mathbf{z}}}{2\sqrt{41}} = \frac{1}{\sqrt{41}}\left(2\,\hat{\mathbf{x}} - 1\,\hat{\mathbf{y}} + 6\,\hat{\mathbf{z}}\right)
\end{aligned}$$

2. *Griffiths 1.3.* Find the angle between the *body* diagonals of a cube. Use a vector product.

Put one corner of the cube at the origin, and let it extend in the region where x, y, z are positive, such that it has vertices at (000), (100), (110), (010), (101), (001), (011), and (111). We could represent two body diagonals by the vectors

$$\begin{aligned}
\vec{\mathbf{A}} &= \hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}} \\
\vec{\mathbf{B}} &= \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}
\end{aligned}$$

Note that for $\vec{\mathbf{A}}$ one should translate the whole vector by 1 unit along $\hat{\mathbf{z}}$ for both diagonals to be within the cube. You should make a sketch to be sure you understand the geometry here. We can use the scalar ("dot") vector product to find the angle θ between the diagonals:

$$\cos\theta = \frac{\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}}{|\vec{\mathbf{A}}||\vec{\mathbf{B}}|} = \frac{1+1-1}{\sqrt{3}\sqrt{3}} = \frac{1}{3} \quad\Longrightarrow\quad \theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.5^\circ$$

3. *GBT Calculus Sect. 5.5, example 2.* If $\vec{\mathbf{a}}=\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}}$, $\vec{\mathbf{b}}=2\hat{\mathbf{x}}-\hat{\mathbf{y}}$, and $\vec{\mathbf{c}}=3\hat{\mathbf{x}}+5\hat{\mathbf{y}}-7\hat{\mathbf{z}}$, verify the identity

$$\vec{\mathbf{a}}\times\left(\vec{\mathbf{b}}\times\vec{\mathbf{c}}\right) = \left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right)\vec{\mathbf{b}} - \left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)\vec{\mathbf{c}}$$

We just need to grind through it. For the left-hand side:

$$\vec{\mathbf{b}} \times \vec{\mathbf{c}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & -1 & 0 \\ 3 & 5 & -7 \end{vmatrix} = 7\hat{\mathbf{x}} + 14\hat{\mathbf{y}} + 13\hat{\mathbf{z}}$$

$$\vec{\mathbf{a}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & -1 & 1 \\ 7 & 14 & 13 \end{vmatrix} = -27\hat{\mathbf{x}} + -6\hat{\mathbf{y}} + 21\hat{\mathbf{z}}$$

For the right-hand side:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{c}} = (\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}) \cdot (3\hat{\mathbf{x}} + 5\hat{\mathbf{y}} - 7\hat{\mathbf{z}}) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 5 & -7 \end{bmatrix} = 3 - 5 - 7 = -9$$

$$(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\, \vec{\mathbf{b}} = -9\vec{\mathbf{b}} = -18\hat{\mathbf{x}} + 9\hat{\mathbf{y}}$$

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \end{bmatrix} = 2 + 1 = 3$$

$$\left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right) \vec{\mathbf{c}} = 3\vec{\mathbf{c}} = 9\hat{\mathbf{x}} + 15\hat{\mathbf{y}} - 21\hat{\mathbf{x}}$$

$$(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\, \vec{\mathbf{b}} - \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right) \vec{\mathbf{c}} = -27\hat{\mathbf{x}} - 6\hat{\mathbf{y}} + 21\hat{\mathbf{z}} = \vec{\mathbf{a}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right)$$

4. *GBT Calculus Sect. 5.4, exercise 16.* If $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are given constant vectors and ω is a constant, describe the trajectory of a particle given by $\vec{\mathbf{r}}(t) = \vec{\mathbf{a}} \cos \omega t + \vec{\mathbf{b}} \sin \omega t$. Verify the following

$$\frac{d^2\vec{\mathbf{r}}}{dt^2} + \omega^2 \vec{\mathbf{r}} = 0 \qquad \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{r}}}{dt} = \omega \vec{\mathbf{a}} \times \vec{\mathbf{b}} \qquad \left|\frac{d\vec{\mathbf{r}}}{dt}\right|^2 + \omega^2 |\vec{\mathbf{r}}|^2 = \omega^2 \left(|\vec{\mathbf{a}}|^2 + |\vec{\mathbf{b}}|^2\right)$$

The trajectory is an ellipse. One can verify this by considering the special case $\vec{\mathbf{a}} = \hat{\mathbf{x}}$, $\vec{\mathbf{b}} = \hat{\mathbf{y}}$ and plotting $\vec{\mathbf{r}}(t)$ or noting

$$\frac{r_x^2}{|\vec{\mathbf{a}}|^2} + \frac{r_y^2}{|\vec{\mathbf{b}}|^2} = 1$$

which is the equation for an ellipse. In order to verify the relationships above, we will need $\frac{d\vec{\mathbf{r}}}{dt}$ and $\frac{d^2\vec{\mathbf{r}}}{dt^2}$; we may as well calculate them now and get it over with.

$$\frac{d\vec{\mathbf{r}}}{dt} = -\omega\vec{\mathbf{a}}\sin\omega t + \omega\vec{\mathbf{b}}\cos\omega t \qquad\qquad \frac{d^2\vec{\mathbf{r}}}{dt^2} = -\omega^2\vec{\mathbf{a}}\cos\omega t - \omega^2\vec{\mathbf{b}}\cos\omega t$$

Let us label the three relationships from left to right as **(i)**, **(ii)**, and **(iii)**
(i): The first relationship is now obvious from the form of $\frac{d^2\vec{\mathbf{r}}}{dt^2}$:

$$\frac{d^2\vec{\mathbf{r}}}{dt^2} = -\omega^2\vec{\mathbf{a}}\cos\omega t - \omega^2\vec{\mathbf{b}}\cos\omega t = -\omega^2\vec{\mathbf{r}} \quad\Longrightarrow\quad \frac{d^2\vec{\mathbf{r}}}{dt^2} + \omega^2\vec{\mathbf{r}} = 0$$

(ii): The second relationship is just conservation of angular momentum (if you multiply the left-hand side by the mass of the particle m). Think about what the answer should be if $\vec{\mathbf{a}}=\hat{\mathbf{x}}$, $\vec{\mathbf{b}}=\hat{\mathbf{y}}$. If we multiply both sides by the particle mass, the left-hand side of the equation is then angular momentum, and the right-hand side is (for a point particle) $mr^2\omega$, or $I\omega$.

One way you will be tempted to solve this is by just defining unit vectors along the directions of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, and then calculating the cross product of each side. This will work ... but it requires subtlety since $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are not necessarily orthogonal, and our methods for calculating cross-products presume the use of an orthogonal set of unit vectors (orthogonal basis). More than likely, you will find that stray factors of $\sin\theta$ have gone missing if you proceed this way.

Anyway: what we need to do is express $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ in terms of a comfortable set of orthogonal unit vectors before proceeding. It may not seem as general to suddenly give up our coordinate-free expressions and pin ourselves to a particular coordinate system, but our result will not be any less valid. Think of it this way: we are just trading the constant vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ for two different constant vectors that happen to be orthogonal. If we wanted to, we could always change to another basis (set of unit vectors) later, or change back to the $\vec{\mathbf{a}}$-$\vec{\mathbf{b}}$ basis, and everything would be fine.

Let us pick $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ as our orthogonal basis. We could just as easily pick spherical coordinates or something else, but this is an easy choice. We choose $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ to define the same plane as $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, which we are also free to do without loss of generality.[i] We may thus decompose $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$

$$\vec{\mathbf{a}} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} \qquad\qquad \vec{\mathbf{b}} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}}$$

[i]So long as $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are not parallel, but the problem would be trivial anyway if that were true.

Therefore,

$$\vec{\mathbf{r}} = (a_x \cos\omega t + b_x \sin\omega t)\,\hat{\mathbf{x}} + (a_y \cos\omega t + b_y \sin\omega t)$$
$$\frac{d\vec{\mathbf{r}}}{dt} = (-\omega a_x \sin\omega t + \omega b_x \cos\omega t)\,\hat{\mathbf{x}} + (-\omega a_y \sin\omega t + \omega b_y \cos\omega t)\,\hat{\mathbf{y}}$$

We can now calculate the cross-product:

$$\begin{aligned}
\vec{\mathbf{r}} \times \frac{d\vec{\mathbf{r}}}{dt} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x \cos\omega t + b_x \sin\omega t & a_y \cos\omega t + b_y \sin\omega t & 0 \\ -\omega a_x \sin\omega t + \omega b_x \cos\omega t & -\omega a_y \sin\omega t + \omega b_y \cos\omega t & 0 \end{vmatrix} \\
&= \hat{\mathbf{z}}\left[(a_x \cos\omega t + b_x \sin\omega t)(-\omega a_y \sin\omega t + \omega b_y \cos\omega t)\right] \\
&\qquad\qquad - \hat{\mathbf{z}}\left[(a_y \cos\omega t + b_y \sin\omega t)(-\omega a_x \sin\omega t + \omega b_x \cos\omega t)\right] \\
&= \omega\hat{\mathbf{z}}\left[a_x b_y - a_y b_x\right] = \omega\vec{\mathbf{a}} \times \vec{\mathbf{b}}
\end{aligned}$$

The term in brackets on the last line is the definition of the cross product of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ in our orthogonal $x-y-z$ basis. Note that since we chose $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ to lie in the $x-y$ plane, their cross product must be along $\hat{\mathbf{z}}$.

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_x b_y - a_y b_x)\,\hat{\mathbf{z}}$$

(iii): Finally, the last relationship follows readily.

$$\begin{aligned}
\left|\frac{d\vec{\mathbf{r}}}{dt}\right|^2 &= \left(-\omega\vec{\mathbf{a}}\sin\omega t + \omega\vec{\mathbf{b}}\cos\omega t\right)\cdot\left(-\omega\vec{\mathbf{a}}\sin\omega t + \omega\vec{\mathbf{b}}\cos\omega t\right) \\
&= \omega^2|\vec{\mathbf{a}}|^2\sin^2\omega t - 2\omega^2\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\sin\omega t\cos\omega t + \omega^2|\vec{\mathbf{b}}|^2\cos^2\omega t \\
\omega^2|\vec{\mathbf{r}}|^2 &= \omega^2\vec{\mathbf{r}}\cdot\vec{\mathbf{r}} = \omega^2\left(|\vec{\mathbf{a}}|^2\cos^2\omega t + 2\omega^2\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\sin\omega t\cos\omega t + |\vec{\mathbf{b}}|^2\sin^2\omega t\right) \\
\left|\frac{d\vec{\mathbf{r}}}{dt}\right|^2 + \omega^2|\vec{\mathbf{r}}|^2 &= \omega^2\left(|\vec{\mathbf{a}}|^2\sin^2\omega t + |\vec{\mathbf{b}}|^2\cos^2\omega t\right) + \omega^2\left(|\vec{\mathbf{a}}|^2\cos^2\omega t + |\vec{\mathbf{b}}|^2\sin^2\omega t\right) = \omega^2\left(|\vec{\mathbf{a}}|^2 + |\vec{\mathbf{b}}|^2\right)
\end{aligned}$$