University of Alabama
Department of Physics and Astronomy

PH 102-1 / LeClair Summer II 2008

Problem Set 2: SOLUTION

1. 10 points. (a) How much negative charge and how much positive charge are there on the electrons and the protons in a cup of water (0.25 kg)? Note Avogadro's number is $N_A = 6.022 \times 10^{23}$, and each oxygen atom has 8 electrons. **(b)** What is the magnitude of the attractive force exerted by the electrons in a cup of water on the protons in a second cup of water at a distance of 10 m?

The "molecular mass" of water is 18 g/mol, so 250 g of water amounts to 250/18 mol. For each mole of water, there are 6.02×10^{23} molecules. Each molecule has two hydrogen atoms (with one electron) and one oxygen atom (with eight electrons), with a total of ten electrons per molecule. Thus, we can calculate the total amount of negative charge in the cup of water readily:

$$\begin{aligned}[\text{neg. chg.}] &= 0.25\,\text{kg}\left[\frac{1000\,\text{g}}{1\,\text{kg}}\right]\left[\frac{1\,\text{mol}}{18\,\text{g}}\right]\left[\frac{6.02\times10^{23}\,\text{molecules}}{\text{mol}}\right]\left[\frac{10\,\text{electrons}}{\text{molecule}}\right]\left[\frac{-1.6\times10^{-19}\,\text{C}}{\text{electron}}\right] \\ &= 1.3\times10^{7}\,\text{C}\end{aligned}$$

Clearly, since the cup of water is overall electrically neutral, the positive charge on the protons is just the opposite of this.

If we treat the total charge in each glass of water as point charges, then in the first cup of water we have one point charge of -1.3×10^{-7} C and another of 1.3×10^{-7} C, separated by 10 m. The force is then:

$$\begin{aligned}|\vec{\mathbf{F}}| &= \frac{k_e q_1 q_2}{r_{12}^2} \\ &= \frac{\left[9\times10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2\right]\left[-1.3\times10^{-7}\,\text{C}\right]\left[+1.3\times10^{-7}\,\text{C}\right]}{\left[10\,\text{m}\right]^2} \\ &= -1.5\times10^{22}\,\text{N}\end{aligned}$$

This is an *enormous* force, equivalent to a weight of a billion billion tons! Thankfully, this attractive force on the protons is precisely canceled by an equally large repulsive force exerted by the protons in one cup on the protons in the other.

2. 5 points. Consider two toner particles separated by 1.2×10^{-5} m; each of the two particles has a negative charge of -30 fC. What is the electric force that one particle exerts on the other? Treat the toner particles approximately as point particles. Note that $1\,\text{fC} = 10^{-15}$ fC.

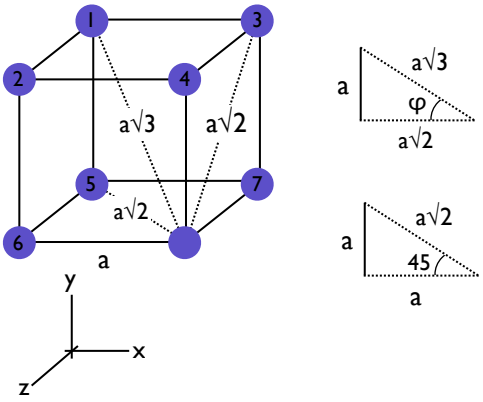
Since we can treat the toner particles as point charges, the force one particle exerts on the other is found easily with Coulomb's law:

$$|\vec{\mathbf{F}}| = \frac{k_e q_1 q_2}{r_{12}^2}$$
$$= \frac{\left[9 \times 10^9\,\mathrm{N \cdot m^2/C^2}\right]\left[-30 \times 10^{-15}\,\mathrm{C}\right]\left[-30 \times 10^{-15}\,\mathrm{C}\right]}{\left[1.2 \times 10^{-5}\,\mathrm{m}\right]^2}$$
$$= 5.6 \times 10^{-8}\,\mathrm{N} = 56\,\mathrm{nN}$$

The positive sign means that the force is repulsive, as it should be for two particles with the same sign charge. The force will tend to push the two particles away from each other, along a line joining them together. This mutual repulsion helps keep the toner particles dispersed. as opposed to clumping together.

3. 10 points. Eight equal charges q are located at the corners of a cube of side a. **(a)** Find the magnitude of the total force on one of the charges due to the other seven charges. **(b)** Find the electric potential at one corner, taking zero potential to be infinitely far away.

Refer to the figure below. Since all charges are equivalent, we can pick any one to be the charge of interest, leaving 7 other charges to consider (numbered). Of those 7 other charges, there are 3 charges a distance a away, 3 a distance $a\sqrt{2}$ away, and one charge a distance $a\sqrt{3}$ away.



By symmetry, the x, y, and z components of the electric force must be equal. Thus, we only need to calculate one component of the total force on the charge of interest. We will choose the coordinate system as indicated in the figure, and calculate the x component of the force on the lower right-most charge. We can already see that several charges will not give an x component of the force at all, just from symmetry - charges 3, 4 and 7. This leaves only charges 1, 2, 5, and 6 to deal with.

Charge 6 will give a force purely in the x direction:

$$F_{6,x} = \frac{k_e q^2}{a^2}$$

Charges 5 and 2 are both a distance $a\sqrt{2}$ away, and a line connecting these charges with the charge of interest makes a $45°$ with the x axis in both cases. Thus, noting that $\cos 45° = 1/\sqrt{2}$,

$$F_{2,x} = F_{5,x} = \frac{k_e q^2}{\left(a\sqrt{2}\right)^2}\cos 45° = \frac{k_e q^2}{2a^2}\frac{1}{\sqrt{2}} = \frac{k_e q^2}{a^2}\frac{1}{2\sqrt{2}}$$

Finally, we have charge 1 to deal with. It is a distance $\sqrt{3}$ away (see the figure). What is the x component of the force from charge 1? First, we can find the component of the force in the $x-z$ plane (see the triangle in the upper right of the figure):

$$F_{1,x-z} = F_1 \cos\varphi = F_1 \frac{\sqrt{2}}{\sqrt{3}}$$

Now, we can find the component of the force along the x direction:

$$F_{1,x} = F_{1,x-z} \cos 45^\circ = F_{1,x-z} \frac{1}{\sqrt{2}} = F_1 \frac{\sqrt{2}}{\sqrt{3}} \frac{1}{\sqrt{2}} = F_1 \frac{1}{\sqrt{3}}$$

Since we know charge 1 is a distance $a\sqrt{3}$ away, we can calculate the full force F_1 easily, and complete the expression for $F_{1,x}$:

$$F_{1,x} = \frac{k_e q^2}{\left(a\sqrt{3}\right)^2} \frac{1}{\sqrt{3}} = \frac{k_e q^2}{a^2} \frac{1}{3\sqrt{3}}$$

Now we have the x component for the force from every charge; the net force in the x direction is just the sum of all those:

$$F_{x,\text{net}} = F_{1,x} + F_{2,x} + F_{5,x} + F_{6,x} = \frac{k_e q^2}{a^2}\left[1 + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}}\right] = \frac{k_e q^2}{a^2}\left[1 + \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{3}}\right]$$

Since the problem is symmetric in the x, y, and z directions, all three components must be equivalent. Thus, the total force is:

$$\vec{\mathbf{F}}_{\text{net}} = \frac{k_e q^2}{a^2}\left[1 + \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{3}}\right][\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}] \approx \frac{k_e q^2}{a^2}[1.90][\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}]$$

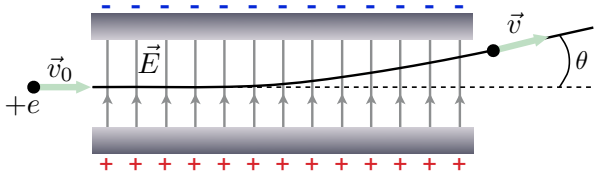
The magnitude of the force is then just

$$|\vec{\mathbf{F}}_{\text{net}}| = \sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{3}\left[1 + \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{3}}\right]\frac{k_e q^2}{a^2} \approx 3.29\frac{k_e q^2}{a^2}$$

The potential is **much** easier to find - potential is a scalar, and we have no components to worry about. All we need to know is that there are 3 charges a distance a away, 3 a distance $a\sqrt{2}$ away, and one charge a distance $a\sqrt{3}$ away. We can find the potential due to each charge separately, and add the results via superposition:

$$V = 3\frac{k_e q}{a} + 3\frac{k_e q}{a\sqrt{2}} + \frac{k_e q}{a\sqrt{3}} = \frac{k_e q}{a}\left[3 + \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right] \approx 5.70\frac{k_e q}{a}$$

4. 10 points. An ion milling machine uses a beam of gallium ions ($m = 70\,\text{u}$) to carve microstructures from a target. A region of uniform electric field between parallel sheets of charge is used for precise control of the beam direction. Single ionized gallium atoms with initially horizontal velocity of $1.8\times10^4\,\text{m/s}$ enter a 2.0 cm-long region of uniform electric field which points vertically upward, as shown below. The ions are redirected by the field, and exit the region at the angle θ shown. If the field is set to a value of $E = 90\,\text{N/C}$, what is the exit angle θ?



A singly-ionized gallium atom has a charge of $q = +e$, and the mass of $m = 70\,\text{u}$ means 70 *atomic mass units*, where one atomic mass unit is $1\,\text{u} = 1.66 \times 10^{-27}\,\text{kg}$.

What we really have here is a particle under the influence of a constant force, just as if we were to throw a ball horizontally and watch its trajectory under the influence of gravity (the only difference is that since we have negative charges, things can "fall up"). To start with, we will place the origin at the ion's initial position, let the positive x axi run to the right, and let the positive y axis run straight up. Thus, the particle starts with a velocity purely in the x direction: $\vec{\mathbf{v}}_0 = v_x\,\hat{\mathbf{x}}$.

While the particle is in the electric-field-containing region, it will experience a force pointing along the $+y$ direction, with a constant magnitude of qE. Since the force acts only in the y direction, there will be a net acceleration only in the y direction, and *the velocity in the x direction will remain constant.* Once outside the region, the particle will experience no net force, and it will therefore continue along in a straight line. It will have acquired a y component to its velocity due to the electric force, but the x component will still be v_x. Thus, the particle exits the region with velocity $\vec{\mathbf{v}} = v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}}$. The angle at which the particle exits the plates, measured with respect to the x axis, must be

$$\tan\theta = \frac{v_y}{v_x}$$

Thus, just like in any mechanics problem, finding the angle is reduced to a problem of finding the final velocity components, of which we already know one. So, how do we find the final velocity in the y direction? Initially, there is no velocity in the y direction, and while the particle is traveling between the plates, there is a net force of qE in the y direction. Thus, the particle experiences an acceleration

$$a_y = \frac{F_y}{m} = \frac{qE_y}{m}$$

The electric field is purely in the y direction in this case, so $E_y = 90\,\text{N/C}$. Now we know the acceleration in the y direction, so if we can find out the time the particle takes to transit the plates, we are done, since the the transit time Δt and acceleration a_y determine v_y:

$$v_y = a_y \Delta t$$

Since the x component of the velocity is not changing, we can find the transit time by noting that the distance covered in the x direction must be the x component of the velocity times the transit time. The distance covered in the x direction is just the width of the plates, so:

$$d_x = v_x \Delta t = 2.0\,\text{cm} \quad \Longrightarrow \quad \Delta t = \frac{d_x}{v_x}$$

Putting the previous equations together, we can express v_y in terms of known quantities:

$$v_y = a_y \Delta t = a_y \frac{d_x}{v_x} = \frac{qE_y}{m}\frac{d_x}{v_x} = \frac{qE_y d_x}{m v_x}$$

Finally, we can now find the angle θ as well:

$$\tan\theta = \frac{v_y}{v_x} = \frac{\frac{qE_y d_x}{mv_x}}{v_x} = \frac{qE_y d_x}{mv_x^2}$$

And that's that. Now we plug in the numbers we have, watching the units carefully:

$$\begin{aligned}
\theta &= \tan^{-1}\left[\frac{qE_y d_x}{mv_x^2}\right] \\
&= \tan^{-1}\left[\frac{\left(1.6\times10^{-19}\,\mathrm{C}\right)\left(90\,\mathrm{N/C}\right)\left(0.02\,\mathrm{m}\right)}{\left(70\cdot1.66\times10^{-27}\,\mathrm{kg}\right)\left(1.8\times10^{4}\,\mathrm{m/s}\right)^2}\right] \\
&= \tan^{-1}\left[7.6\times10^{-3}\,\frac{\mathrm{N}}{\mathrm{kg\cdot m/s^2}}\right] \qquad \text{note } 1\,\mathrm{N} = 1\,\mathrm{kg\cdot m/s^2} \\
&= \tan^{-1} 7.6\times10^{-3} \\
&\approx 0.44^\circ
\end{aligned}$$

5. 5 points. A sphere the size of a basketball is charged to a potential of $-1000\,\mathrm{V}$. About how many extra electrons are on it, per cm^2 of surface?

If the charge is spread evenly over the surface of a spherical object, like a basketball, then Gauss' law says we may treat the charge distribution as a point charge. Thus, we may consider the uniformly charged basketball equivalent to a single point charge at the center of the basketball. A men's regulation basketball has a diameter of 9.39 in, and thus a radius of 0.12 m. We know then that the potential at the surface of the basketball is $-1000\,\mathrm{V}$, and this potential results from an effective point charge q at the center of the basketball, 0.12 m from the surface. Using the expression for the potential from a point charge:

$$\begin{aligned}
-1000\,\mathrm{V} &= \frac{k_e q}{r} = \frac{\left[9\times10^{9}\,\mathrm{N\cdot m^2/C^2}\right]q}{0.12\,\mathrm{m}} \\
\implies \quad q &= -1.33\times10^{-8}\,\mathrm{C}
\end{aligned}$$

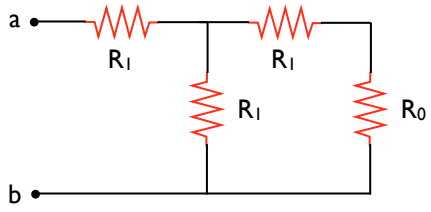
This is the effective charge on the basketball. Given that one electron has $-1.6\times10^{-19}\,\mathrm{C}$ of charge, this must correspond to:

$$\text{(number of electrons)} = \frac{-1.33\times10^{-8}\,\mathrm{C}}{-1.6\times10^{-19}\,\mathrm{C/electron}} = 8.33\times10^{10}\,\text{electrons}$$

If these electrons are spread out evenly over the surface, the electron density can be calculated from the surface area of the basketball (remembering that the surface area of a sphere is $4\pi r^2$, and we were asked to use cm^2):

$$\text{(density of electrons)} = \frac{8.33\times10^{10}\,\text{electrons}}{4\pi\left(12\,\mathrm{cm}\right)^2} \approx 4.6\times10^{7}\,\text{electrons/cm}^2$$

6. 15 points. In the circuit below, if R_0 is given, what value must the R_1 have for the equivalent resistance between the two terminals a and b to be R_0?



This one is, admittedly, a bit messy. The end result does have a certain elegance though ...

With any complicated resistor problem, we first try to find sets of two resistors purely in parallel or purely in series. Combine any such pairs, lather, rinse, repeat. The first pair we can spot - and the only one which is purely in series or parallel - is resistor R_0 in series with the rightmost R_1. We cannot combine any other resistors, since no other pairs are purely in series or parallel. Putting together R_1 and R_0 makes an equivalent resistor R_2, whose value we can calculate easily:

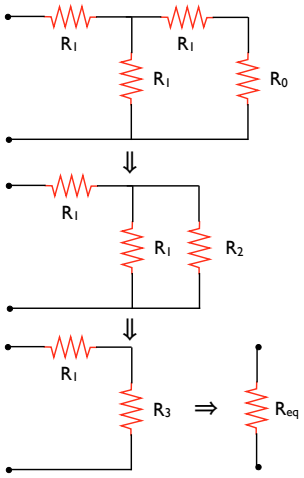
$$R_2 = R_1 + R_0$$

This will leave the new resistor purely in *parallel* with the middle R_1, which means we can combine R_2 and R_1 into a new resistor R_3:

$$\frac{1}{R_3} = \frac{1}{R_2} + \frac{1}{R_1} = \frac{1}{R_1 + R_0} + \frac{1}{R_1} = \frac{R_1 + R_0 + R_1}{R_1\left(R_1 + R_0\right)} = \frac{2R_1 + R_0}{R_1^2 + R_1 R_0}$$

$$\implies \quad R_3 = \frac{R_1 R_0 + R_1^2}{2R_1 + R_0}$$

Our progress so far is shown below.



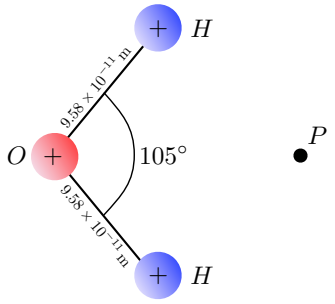
Now we only have R_3 and one R_1 left, purely in series. Combining them will give us one single equivalent resistor R_{eq}:

$$\begin{aligned} R_{eq} = R_1 + R_3 &= \frac{R_1 R_0 + R_1^2}{2R_1 + R_0} + R_1 = \frac{R_1 R_0 + R_1^2}{2R_1 + R_0} + \frac{R_1\left(2R_1 + R_0\right)}{2R_1 + R_0} \\ &= \frac{R_1 R_0 + R_1^2 + 2R_1^2 + R_1 R_0}{2R_1 + R_0} \\ &= \frac{3R_1^2 + 2R_1 R_0}{2R_1 + R_0} \end{aligned}$$

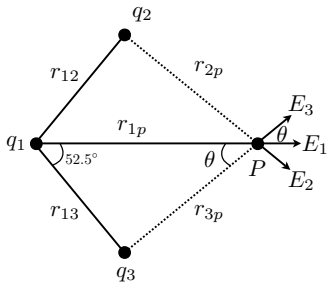
The final bit of the problem says that we want the equivalent resistance to be exactly R_0. We just need to set the above equal to R_0, and solve for R_1 in terms of R_0.

$$\begin{aligned} R_0 &= \frac{3R_1^2 + 2R_1R_0}{2R_1 + R_0} \\ R_0\,(2R_1 + R_0) &= 3R_1^2 + 2R_1R_0 \\ 2R_0R_1 + R_0^2 &= 3R_1^2 + 2R_1R_0 \\ R_0^2 &= 3R_1^2 \\ \implies \quad R_1 &= \frac{R_0}{\sqrt{3}} \end{aligned}$$

7. 10 points. The distance between the oxygen nucleus and each of the hydrogen nuclei in an H_2O molecule is 9.58×10^{-11} m, and the bond angle between hydrogen atoms is 105°. **(a)** Find the electric field produced by the nuclear charges (positive charges) at the point P a distance 1.2×10^{-10} m to the right of the oxygen nucleus. **(b)** Find the electric potential at P.



First, we need to define the geometry of the situation a bit more clearly, and label things properly. Have a look:



Rather than worry about which nucleus is which, we will simply label the charges q_1, q_2, and q_3 and be as general as possible. We will also label the distances in a generic but self-explanatory way: the distance from charge 1 to charge 2 is r_{12}, the distance from charge 3 to the point P is r_{3p}, and so on.

First, connect q_1 and P with a straight line. This is our x axis, and it nicely splits the problem into two symmetric halves. Since the bond angle was given as 105°, we know that the angle $\angle Pq_1q_3$ must be 52.5°, as must the angle $\angle Pq_1q_2$. The electric field due to charge 1 will clearly point directly along the x axis toward point P. The electric field due to charge 3 will make an angle θ with the x axis. Clearly, by symmetry, since $q_3 = q_2$ the electric fields from charges 2 and 3 will have the same x components, but equal and opposite y components - $E_{2x} = E_{3x}$, $E_{2y} = -E_{3y}$. Thus, the fields from charges 2 and 3 will

in total have only an x component - so it is enough to compute only the x component of the field. And, since the x components are the same, we really only need to find one of them. In total, the field at P is then only composed of x components, and requires only two calculations:

$$\vec{\mathbf{E}}_P = [E_{2x} + E_{3x} + E_1]\ \hat{\mathbf{x}} = [2E_{3x} + E_1]\ \hat{\mathbf{x}}$$

First, we can easily find E_1, since we are told $r_{1p}{=}1.2{\times}10^{-10}\,\mathrm{m}$:

$$E_1 = \frac{k_e q_1}{r_{1p}^2}$$

In order to find E_{3x}, we need two things: the angle θ, and the distance of charge 3 to point P, viz., r_{3p}. We can find the latter in terms of known quantities using the law of cosines[i] on the triangle $\triangle q_1 P q_3$ with the 52.5° angle

$$r_{3p}^2 = r_{1p}^2 + r_{13}^2 - 2r_{13}r_{1p}\cos 52.5^\circ \approx 9.79 \times 10^{-11}\,\mathrm{m}$$

Once we have r_{3p}, we can find the angle θ by using the law of cosines on the same triangle, this time about the angle θ:

$$\begin{aligned} r_{13}^2 &= r_{1p}^2 + r_{3p}^2 - 2r_{1p}r_{3p}\cos\theta \\ \implies \quad \cos\theta &= \frac{r_{1p}^2 + r_{3p}^2 - r_{13}^2}{2r_{1p}r_{3p}} \approx 0.631 \\ \implies \quad \theta &\approx 50.9^\circ \end{aligned}$$

Once we have the angle and distance, we can easily find E_3, and then its x component:

$$\begin{aligned} E_3 &= \frac{k_3 q_3}{r_{3p}^2} \\ E_{3x} &= E_3\cos\theta = \frac{k_3 q_3}{r_{3p}^2}\cos\theta \end{aligned}$$

Since the x component of the field from charge 2 is the same (and the y components of E_2 and E_3 cancel), we are ready to find the total field at point P:

$$\vec{\mathbf{E}}_P = [2E_{3x} + E_1]\ \hat{\mathbf{x}} = \left[2\left(\frac{k_3 q_3}{r_{3p}^2}\cos\theta\right) + \frac{k_e q_1}{r_{1p}^2}\right]\ \hat{\mathbf{x}} \approx \left[9.9 \times 10^{11}\,\mathrm{V/m}\right]\ \hat{\mathbf{x}}$$

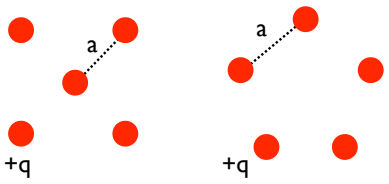
Now, when you get to the point of actually plugging in numbers, remember: the charge on a hydrogen nucleus, with a single proton, is $+e$, while that on an oxygen nucleus is $+8e$.

What about the potential at point P? Far easier, no vectors! We have two charges a distance r_{3p} away, and one a distance r_{1p} away (again, we know that the contributions from charges 2 and 3 will be the same):

$$V_P = \frac{k_e q_1}{r_{1p}} + \frac{k_e q_2}{r_{2p}} + \frac{k_e q_3}{r_{3p}} = \frac{k_e q_1}{r_{1p}} + 2\frac{k_e q_3}{r_{3p}} \approx 125\,\mathrm{V}$$

[i] This is a very useful trick, and remembering if you have forgotten. `http://en.wikipedia.org/wiki/Law_of_cosines`.

8. 15 points. Five identical point charges $+q$ are arranged in two different manners as shown below - in once case as a face-centered square, in the other as a regular pentagon. Find the potential energy of each system of charges, taking the zero of potential energy to be infinitely far away. Express your answer in terms of a constant times the energy of two charges $+q$ separated by a distance a.



Using the principle of superposition, we know that the potential energy of a system of charges is just the sum of the potential energies for all the unique pairs of charges. The problem is then reduced to figuring out how many different possible pairings of charges there are, and what the energy of each pairing is. The potential energy for a single pair of charges, both of magnitude q, separated by a distance d is just:

$$PE_{\text{pair}} = \frac{k_e q^2}{d}$$

Since all of the charges are the same in both configurations, all we need to do is figure out how many pairs there are in each situation, and for each pair, how far apart the charges are.

How many unique pairs of charges are there? There are not so many that we couldn't just list them by brute force - which we will do as a check - but we can also calculate how many there are. In both configurations, we have 10 charges, and we want to choose all possible groups of 2 charges that are not repetitions. So far as potential energy is concerned, the pair $(2,1)$ is the same as $(1,2)$. Pairings like this are known as combinations, as opposed to *permutations* where $(1,2)$ and $(2,1)$ are *not* the same. Calculating the number of possible combinations is done like this:[ii]

$$\text{ways of choosing pairs from five charges} = \binom{5}{2} = {}^5C_2 = \frac{5!}{2!\,(5-2)!} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 1\cdot 3\cdot 2\cdot 1} = 10$$

So there are 10 unique ways to choose 2 charges out of 5. First, let's consider the face-centered square lattice. In order to enumerate the possible pairings, we should label the charges to keep them straight. Label the corner charges $1-4$, and the center charge 5 (it doesn't matter which way you number the corners, just so long as 5 is the middle charge). Then our possible pairings are:

$(1,2)$ $(1,3)$ $(1,4)$ $(1,5)$
$(2,3)$ $(2,3)$ $(2,5)$
$(3,4)$ $(3,5)$
$(4,5)$

And there are ten, just as we expect. In this configuration, there are only three different distances that can separate a pair of charges: pairs on adjacent corners are a distance $a\sqrt{2}$ apart, a center-corner pairing is a distance a apart, and a far corner-far corner pair is $2a$ apart. We can take our list above, and sort it into pairs that have the same separation:

[ii] A nice discussion of combinations and permutations is here: `http://www.themathpage.com/aPreCalc/permutations-combinations.htm`

Table 1: *Charge pairings in the square lattice*

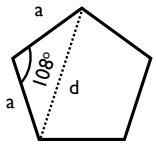
#, pairing type	separation				pairs
4, center-corner	a	$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$
4, adjacent corners	$a\sqrt{2}$	$(1,4)$	$(3,4)$	$(2,3)$	$(1,2)$
2, far corner	$2a$			$(1,3)$	$(2,4)$

And we are nearly done already. We have four pairs of charges a distance a apart, four that are $a\sqrt{2}$ apart, and two that are $2a$ apart. Write down the energy for each type of pair, multiply by the number of those pairs, and add the results together:

$$\begin{aligned} PE_{\text{square}} &= 4\,(\text{center-corner pair}) + 2\,(\text{far corner pair}) + 4\,(\text{adjacent corner pair}) \\ &= 4\left[\frac{k_e q^2}{a}\right] + 2\left[\frac{k_e q^2}{2a}\right] + 4\left[\frac{k_e q^2}{a\sqrt{2}}\right] \\ &= \frac{k_e q^2}{a}\left[4 + 1 + \frac{4}{\sqrt{2}}\right] \\ &= \frac{k_e q^2}{a}\left[5 + 2\sqrt{2}\right] \approx 7.83\frac{kq^2}{a} \end{aligned}$$

For the pentagon lattice, things are even easier. This time, just pick one charge as "1", and label the others from 2-5 in a clockwise or counter-clockwise fashion. Since we still have 5 charges, there are still 10 pairings, and they are the same as the list above. For the pentagon, however, there are only *two* distinct distances - either charges can be adjacent, and thus a distance a apart, or they can be next-nearest neighbors. What is the next-nearest neighbor distance?

In a regular pentagon, each of the angles is 108°, and in our case, each of the sides has length a, as shown below. We can use the law of cosines to find the distance d between next-nearest neighbors.



$$\begin{aligned} & d^2 = a^2 + a^2 - 2\cdot a\cdot a\cos 108^\circ = 2a^2\,(1-\cos 108^\circ) \\ \Longrightarrow \quad & d = a\sqrt{2-2\cos 108^\circ} = a\phi \approx 1.618a \end{aligned}$$

Here the number ϕ is known as the "Golden Ratio." The distances a and d automatically satisfy the golden ratio in a regular pentagon, $d/a = \phi$. Given the nearest neighbor distance in terms of a, we can then create a table of pairings for the pentagon (Table 2).

Table 2: *Charge pairings in the pentagonal lattice*

#, pairing type	separation					pairs
5, next-nearest neighbors	d	$(1,3)$	$(1,4)$	$(2,4)$	$(2,5)$	$(3,5)$
5, adjacent	a	$(1,2)$	$(2,3)$	$(3,4)$	$(4,5)$	$(5,1)$

Now once again we write down the energy for each type of pair, and multiply by the number of pairs:

$$\begin{aligned}
PE_{\text{pentagon}} &= 5\,(\text{energy of adjacent pair}) + 5\,(\text{energy of next-nearest neighbor pair}) \\
&= 5\left[\frac{k_e q^2}{a}\right] + 5\left[\frac{k_e q^2}{d}\right] \\
&= 5\left[\frac{k_e q^2}{a}\right] + 5\left[\frac{k_e q^2}{a\sqrt{2-2\cos 108^\circ}}\right] \\
&= \frac{k_e q^2}{a}\left[5 + \frac{5}{\sqrt{2-2\cos 108^\circ}}\right] \\
&\approx \frac{k_e q^2}{a}\left[5 + \frac{5}{1.618}\right] \approx 8.09\frac{kq^2}{a}
\end{aligned}$$

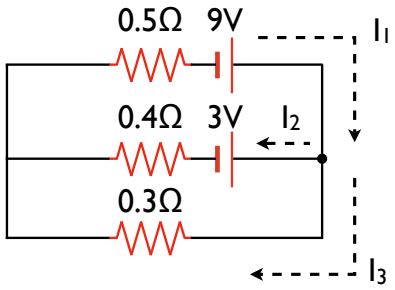
So the energy of the pentagonal lattice is higher, meaning it is less favorable than the square lattice. Neither one is energetically favored though - since the energy is positive, it means that either configuration of charges is less stable than just having all five charges infinitely far from each other. This makes sense - if all five charges have the same sign, they don't want to arrange next to one another, and thus these arrangements cost energy to keep together. If we didn't force the charges together in these patterns, the positive energy tells us that they would fly apart given half a chance. For this reason, neither one is a valid sort of crystal lattice, real crystals have equal numbers of positive and negative charges, and are overall electrically neutral.

9. 10 points. You are given two batteries, one of 9 V and internal resistance 0.50 Ω, and another of 3 V and internal resistance 0.40 Ω. How must these batteries be connected to give the largest possible current through an external 0.30 Ω resistor? What is this current?

There are basically two interesting ways to hook up the components given: all series, and all parallel. First, one can put everything in series. In series, the circuit is simple. You have three resistors and two batteries, and since there is only a single current in the circuit, which we'll call I, you can readily add up the voltage drops around the circuit to find I:

$$\begin{aligned}
\text{series:} \quad -0.5\,\Omega\, I + 9\,\text{V} - 0.4\,\Omega\, I + 3\,\text{V} - 0.3\,\Omega\, I &= 0 \\
12\,\text{V} - 1.2\,\Omega\, I &= 0 \\
I &= 10\,\text{A}
\end{aligned}$$

Putting everything in parallel looks like this:



In this case, there are three currents to deal with, it is the third I_3 that we are interested in. First, we can apply the "junction rule" at the circular dot on the right-hand side of the circuit. Current I_1 enters the junction, currents I_2 and I_3 leave:

$$I_1 = I_2 + I_3$$

Next, we can apply the "loop rule" around the upper-most loop, going clockwise. Remember that crossing a battery from the little pole (-) to the big pole (+) is a *gain* in voltage.

$$-0.5\,\Omega\, I_1 + 9\,\mathrm{V} - 3\,\mathrm{V} - 0.4\,\Omega\, I_2 = 0$$

We can do the same for the lower-most loop:

$$-0.4\,\Omega\, I_2 + 3\,\mathrm{V} - 0.3\,\Omega I_3 = 0$$

Summarizing our three equations so far (and dropping the units):

$$\begin{aligned} I_1 - I_2 - I_3 &= 0 \\ -0.5I_1 - 0.4I_2 &= -6 \\ 0.4I_2 - 0.3I_3 &= -3 \end{aligned}$$

We now have three equations and three unknowns. There are a few ways to go about solving them, I will illustrate two. First, plug the first equation into the third, and solve that for I_1

$$\begin{aligned} & 0.4I_2 - 0.3\,(I_1 - I_2) = 0.7I_2 - 0.3I_1 = -3 \\ \Longrightarrow \quad & I_1 = \frac{0.7}{0.3} I_2 + \frac{3}{0.3} \end{aligned}$$

Now plug that into the second equation we have:

$$\begin{aligned} -0.5I_1 - 0.4I_2 &= -0.5\left(\frac{0.7}{0.3}\right) - 0.4\left(\frac{3}{0.3}\right) - 0.4I_2 = -6 \\ I_2\left(0.4 + 0.5\frac{0.7}{0.3}\right) &= 6 - 0.5\left(\frac{3}{0.3}\right) \\ I_2 &= 0.638\,\mathrm{A} \end{aligned}$$

Now that we have I_2, we can use the third equation to find I_3, the desired current through the $0.3\,\Omega$ resistor:

$$I_3 = \frac{0.4I_2 + 3}{0.3} = 10.85\,\mathrm{A}$$

Thus, connecting everything in parallel gives a slightly higher current through the resistor. One could also try to put two components in series and the third in parallel with that; you can quickly verify that none of those three combinations yield a larger current.

Another way to solve this, perhaps more quickly, is to use matrices and Cramer's rule,[iii] if you are familiar with this technique. If you are not familiar with matrices, you can skip to the next problem - you are not required or necessarily expected to know how to do this. First, write the three equations in matrix form:

[iii] See 'Cramer's rule' in the Wikipedia to see how this works.

$$\begin{bmatrix} 1 & -1 & -1 \\ -0.5 & -0.4 & 0 \\ 0 & 0.4 & -0.3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ -3 \end{bmatrix}$$
$$\mathbf{aI} = \mathbf{V}$$

The matrix **a** times the column vector **I** gives the column vector **V**, and we can use the determinant of the matrix **a** with Cramer's rule to find the currents. For each current, we construct a new matrix, which is the same as the matrix **a** except that the the corresponding column is replaced the column vector **V**. Thus, for I_1, we replace column 1 in **a** with **V**, and for I_2, we replace column 2 in **a** with **V**. We find the current then by taking the new matrix, calculating its determinant, and dividing that by the determinant of **a**. Below, we have highlighted the columns in **a** which have been replaced to make this more clear:

$$I_1 = \frac{\begin{vmatrix} 0 & -1 & -1 \\ -6 & -0.4 & 0 \\ -3 & 0.4 & -0.3 \end{vmatrix}}{\det \mathbf{a}} \qquad I_2 = \frac{\begin{vmatrix} 1 & 0 & -1 \\ -0.5 & -6 & 0 \\ 0 & -3 & -0.3 \end{vmatrix}}{\det \mathbf{a}} \qquad I_3 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ -0.5 & -0.4 & -6 \\ 0 & 0.4 & -3 \end{vmatrix}}{\det \mathbf{a}}$$

Now we need to calculate the determinant of each new matrix, and divide that by the determinant of **a**.[iv] First, the determinant of **a**.

$$\det a = (1)(-0.4)(-0.3) - (1)(0)(0.4) + (-1)(0)(0) - (-1)(-0.5)(-0.3) + (-1)(-0.5)(0.4) - (-1)(-0.4)(0) = 0.47$$

We can now find the currents readily from the determinants of the modified matrices above and that of **a** we just found. We really only want I_3, so we can find that directly:

$$I_3 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ -0.5 & -0.4 & -6 \\ 0 & 0.4 & -3 \end{vmatrix}}{\det \mathbf{a}} = \frac{3(0.4) + 6(0.4) + 3(0.5)}{0.47} = 10.85\,\text{A}$$

This time, we omitted the terms in the determinant which give zeros. Once you are familiar with this method of solving systems of equations, it can be quite efficient. You can complete the same procedure for I_2 and I_1, you should find $I_2 = 0.638\,\text{A}$ and $I_1 = 11.49\,\text{A}$.

10. 10 points. Two capacitors, one charged and the other uncharged, are connected in parallel. **(a)** Prove that when equilibrium is reached, each carries a fraction of the initial charge equal to the ratio of its capacitance to the sum of the two capacitances. **(b)** Show that the final energy is less than the initial energy, and derive a formula for the difference in terms of the initial charge and the two capacitances.

This problem is easiest to start if you approach it from a conservation of energy & charge point of view. We have two capacitors. Initially, one capacitor stores a charge Q_{1i}, while the other is empty, $Q_{2i} = 0$. After connecting them together in parallel, some charge leaves the first capacitor and goes to the second, leaving the two with charges Q_{1f} and Q_{2f}, respectively. Now, since there were no sources hooked up, and we just have the two capacitors, the total amount of charge must be the same before and after we hook them together:

[iv] Again, the Wikipedia entry for 'determinant' is quite instructive.

$$\begin{aligned} Q_i &= Q_f \\ Q_{1i} + Q_{2i} &= Q_{1f} + Q_{2f} \\ Q_{1i} &= Q_{1f} + Q_{2f} \end{aligned}$$

We also know that if two capacitors are connected in parallel, they will have the same voltage ΔV across them:

$$\Delta V_f = \frac{Q_{1f}}{C_1} = \frac{Q_{2f}}{C_2}$$

The fraction of the total charge left on the first capacitor can be found readily combining what we have:

$$\frac{Q_{1f}}{Q_i} = \frac{Q_{1f}}{Q_{1i}} = \frac{Q_{1f}}{Q_{1f} + Q_{2f}} = \frac{Q_{1f}}{Q_{1f} + \frac{C_2}{C_1} Q_{1f}} = \frac{C_1 Q_{1f}}{C_1 Q_{1f} + C_2 Q_{1f}} = \frac{C_1}{C_1 + C_2}$$

The second capacitor must have the rest of the charge:

$$\frac{Q_{2f}}{Q_i} = 1 - \frac{C_1}{C_1 + C_2} = \frac{C_2}{C_1 + C_2}$$

That was charge conservation. We can also apply energy conservation, noting that the energy of a charged capacitor is $Q^2/2C$:

$$\begin{aligned} E_i &= E_f \\ \frac{Q_{1i}^2}{2C_1} &= \frac{Q_{1f}^2}{2C_1} + \frac{Q_{2f}^2}{2C_2} \end{aligned}$$

The final energy can be simplified using the result of the first part of the problem - we note that $Q_{1f} = Q_i C_1 / (C_1 + C_2)$ and $Q_{2f} = Q_i C_2 / (C_1 + C_2)$

$$\begin{aligned} E_f &= \frac{Q_{1f}^2}{2C_1} + \frac{Q_{2f}^2}{2C_2} \\ &= \left(\frac{Q_i C_1}{C_1 + C_2} \right)^2 \frac{1}{2C_1} + \left(\frac{Q_i C_2}{C_1 + C_2} \right)^2 \frac{1}{2C_2} \\ &= \frac{Q_i^2 C_1}{2 (C_1 + C_2)^2} + \frac{Q_i^2 C_2}{2 (C_1 + C_2)^2} \\ &= \frac{Q_i^2 (C_1 + C_2)}{2 (C_1 + C_2)^2} = \frac{Q_i^2}{2 (C_1 + C_2)} \\ &= \frac{Q_i^2}{2C_1} \left(\frac{C_1}{C_1 + C_2} \right) = E_i \left(\frac{C_1}{C_1 + C_2} \right) \end{aligned}$$

Thus, the final energy will be less than the initial energy, by a factor $C_1 / (C_1 + C_2) < 1$.