

UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

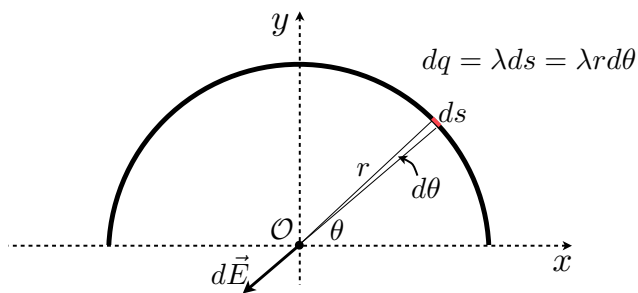
PH 126 / LeClair

Fall 2009

Problem Set 2: Solutions

1. *Purcell 1.5* A thin plastic rod bent into a semicircle of radius r has a charge of Q , in coulombs, distributed uniformly over its length. Find the strength of the electric field at the center of the semicircle.

This is easiest if we use a cartesian coordinate system with its origin at the center of the semicircle. We want the field at the origin. By symmetry, it must be pointing purely downward, but we'll calculate both horizontal and vertical components anyway. Have a look at the figure:



We can break the whole semicircle up into infinitesimal bits of arclength ds , each of which is then a point charge. If one bit is defined by subtending an angle $d\theta$ over a radius r , then we know the arclength is $ds = r d\theta$. This tiny bit of the semicircle carries a net charge dq . If the whole semicircle has a charge Q , then the charge per unit length is just $\lambda = Q/\pi r$, since the length of the plastic is πr . That means that each bit ds has a charge $dq = \lambda ds = \lambda r d\theta$. We can calculate the field from this point charge easily:

$$d\vec{E} = \frac{k dq}{r^2} \hat{r} = \frac{k \lambda r d\theta}{r^2} \hat{r} = \frac{k \lambda d\theta}{r} \hat{r} \quad (1)$$

We know the field will be purely in the $-y$ direction by symmetry, so we really just need to calculate the y component and we are done. For that, we need to multiply by $\sin \theta$, with a minus sign to keep the direction straight:

$$dE_y = -\frac{k \lambda}{r} \sin \theta d\theta \quad (2)$$

That's the field from one little dq . We find the total field from the whole semicircle by integrating over all such dq , which means running our angle θ from $0 \rightarrow \pi$. The radius is fixed in this case, so θ is our only integration variable. Nice!

$$E_y = \int_0^\pi -\frac{k \lambda}{r} \sin \theta d\theta = \frac{k \lambda}{r} \cos \theta \Big|_0^\pi = \frac{-2k \lambda}{r} = \frac{-2kQ}{\pi r^2} = \frac{-Q}{2\pi^2 \epsilon_0 r^2} \quad (3)$$

For the last part, we used our definition $\lambda = Q/\pi r$. The total field is then $\vec{E} = E_y \hat{y}$. Are we sure the x component cancels? If you don't believe in symmetry, believe in integration. All we need to do is replace sin with cos above ...

$$dE_x = -\frac{k\lambda}{r} \cos \theta \, d\theta \quad (4)$$

$$E_x = \int_0^{2\pi} -\frac{k\lambda}{r} \cos \theta \, d\theta = -\frac{k\lambda}{r} \sin \theta \Big|_0^{2\pi} = 0 \quad (5)$$

2. *Purcell 1.11* A charge of 1 C sits at the origin. A charge of -2 C is at $x=1$ on the x axis.

(a) Find a point on the x axis where the electric field is zero.

(b) Locate, at least approximately, a point on the y axis where the electric field is parallel to the x axis.

Let us be a bit more general and show our contempt for mere numbers. Consider a charge q_1 at the origin and a charge q_2 at position $x_0 > 0$ on the x axis. We will assume $|q_2| > |q_1|$, $q_1 > 0$ and $q_2 < 0$.

First, we need to do the physics before we get to any math. Where *could* the field be zero along the x axis? In the region between the two charges, the positive charge q_1 gives a field in the \hat{x} direction, and the negative charge q_2 *also* gives a field in the \hat{x} direction. Since both fields act in the same direction, they cannot possibly cancel each other.

How about for $x > x_0$ (i.e., to the right of the negative charge)? Well, in this region we are farther from q_1 than we are q_2 , so the field from q_1 is smaller than that of q_2 . Further, if $|q_2| > |q_1|$, the field of q_1 is smaller to start with, even at the same distance. Even though the fields are in opposite directions, since they decay as $1/x^2$ and q_1 is smaller and farther away, the two fields cannot possibly cancel each other.

We are left with $x < 0$. This makes sense: the two fields are in opposite directions, and we are closer to the smaller charge. The smaller field at a given distance can be compensated by simply getting closer to the smaller charge. Let us consider a position $-x$ along the x axis, which means we are a distance x from q_1 and $x + x_0$ from q_2 . The total field is then:

$$E = E_1 + E_2 = \frac{kq_1}{x^2} + \frac{kq_2}{(x + x_0)^2} \quad (6)$$

Keep in mind q_2 is negative. We desire $E=0$. Thus,

$$\frac{kq_1}{x^2} = -\frac{kq_2}{(x + x_0)^2} \quad (7)$$

Since we already know our point of interest along the negative x axis, we can just take the square root of both sides and solve this thing quickly – we don't need to worry about the \pm since we already figured that part out. Thus,

$$\begin{aligned}\frac{x}{\sqrt{q_1}} &= \frac{x + x_o}{\sqrt{-q_2}} \\ \frac{x_o}{\sqrt{-q_2}} &= x \left(\frac{1}{\sqrt{q_1}} - \frac{1}{\sqrt{-q_2}} \right) \\ x &= \frac{x_o}{\frac{\sqrt{-q_2}}{\sqrt{q_1}} - 1} = \frac{x_o \sqrt{q_1}}{\sqrt{-q_2} - \sqrt{-q_1}}\end{aligned}\quad (8)$$

With the numbers given,

$$x = \frac{1}{\sqrt{2} - 1} = -\sqrt{2} - 1 \approx -2.4 \text{ m} \quad (9)$$

For the next part, we want a point along the y axis where the field is purely in the horizontal direction. If that is to be true, then we must find a point where the y components of the field from each charge exactly balance. From the charge q_1 this is easy: if we are a distance y along the y axis, then

$$E_{1y} = \frac{kq_1}{y^2} \quad (10)$$

The charge q_2 is at a distance $\sqrt{x_o^2 + y^2}$ from the same point on the y axis. Its field in the y direction is then found similarly, accounting for the geometric factor to pick out the y component.

$$E_{2y} = \frac{kq_2}{x_o^2 + y^2} \frac{x_o}{\sqrt{x_o^2 + y^2}} = \frac{kq_2 y}{(x_o^2 + y^2)^{3/2}} \quad (11)$$

We need $E_{1y} = -E_{2y}$ for the net vertical field to be zero (resulting in a purely horizontal field):

$$E_{1y} = -E_{2y} \quad (12)$$

$$\frac{kq_1}{y^2} = \frac{-kq_2 y}{(x_o^2 + y^2)^{3/2}} \quad (13)$$

$$q_1 (x_o^2 + y^2)^{3/2} = -q_2 y^3 \quad (14)$$

$$q_1^{2/3} (x_o^2 + y^2) = (-q_2)^{2/3} y^2 \quad (15)$$

$$y^2 \left(q_1^{2/3} - (-q_2)^{2/3} \right) = q_1^{2/3} x_o^2 \quad (16)$$

$$y = \frac{\pm q_1^{1/3} x_o}{\sqrt{(-q_2)^{2/3} - q_1^{2/3}}} = \frac{\pm 1}{\sqrt{2^{2/3} - 1}} \approx \pm 1.30 \text{ m} \quad (17)$$

Clearly, by symmetry there are two equivalent points along the y axis, so our \pm makes sense here.

3. *Purcell 1.21* The neutral hydrogen atom in its normal state behaves in some respects like an electric charge distribution which consists of a point charge of magnitude e surrounded by a distribution of negative charge whose density is given by

$$-\rho(r) = Ce^{-2r/a_0}$$

Here a_0 is the *Bohr radius*, 0.53×10^{-10} m, and C is a constant with the value required to make the total amount of negative charge exactly e .

- (a) What is the net electric charge inside a sphere of radius a_0 ?
- (b) What is the electric field strength at this distance from the nucleus?
- (c) What is C ?

Charge is charge density integrated through a volume, just as mass is density integrated through volume:

$$q = \int \rho \, dV \tag{18}$$

If we want the charge contained in a radius a_0 from the origin, we simply perform the integral over the interval $r : 0 \rightarrow a_0$ and over the full range of θ and ϕ . Performing the integral over all space (i.e., $r : 0 \rightarrow \infty$) would give us the whole charge of an electron, which must be $-e$. We'll use this fact in part c.

The first question is: how to perform the volume integral? Our function is radially symmetric, and this strongly suggests – no, *insists* – that we use spherical coordinates. Let's try setting it up in cartesian coordinates to see why. In cartesian coordinates, a volume element is $dV = dx \, dy \, dz$, and our radius is $r = \sqrt{x^2 + y^2 + z^2}$. Thus,

$$q = \int dx \int dy \int dz - Ce^{-\frac{2}{a_0} \sqrt{x^2 + y^2 + z^2}} dx \, dy \, dz \tag{19}$$

where the limits of integration correspond to $\sqrt{x^2 + y^2 + z^2}$ running from 0 to a_0 . This integral puzzles even the Wolfram Integrator ... which is not a good sign. In spherical coordinates, our volume element is

$$dV = r^2 \sin \theta \, dr \, d\theta \, d\phi \quad \text{where} \quad r : 0 \rightarrow a_0; \theta : 0 \rightarrow \pi; \phi : 0 \rightarrow 2\pi \tag{20}$$

where we have also indicated the appropriate angular ranges for the integration. What is nice about this? First, there are no θ or ϕ terms in our function, so those integrals are trivial. Second, the integral over r is tractable. Let's grind through it:

$$q(a_0) = \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\phi \int_0^{a_0} \rho(r) r^2 \, dr \tag{21}$$

$$\text{note} \quad \int_0^\pi \sin \theta \, d\theta = -\cos \theta \Big|_0^\pi = 2 \quad \text{and} \quad \int_0^{2\pi} d\phi = \phi \Big|_0^{2\pi} = 2\pi \tag{22}$$

$$q(a_0) = 4\pi \int_0^{a_0} \rho(r) r^2 \, dr \tag{23}$$

The last integral is doable. You can integrate by parts twice, starting with $u = r$ and $dv = e^{-2r/a_0}$. Also, Wolfram

is all over it. The result is

$$q(a_o) = -4\pi C \int_0^{a_o} r^2 e^{-2r/a_o} dr = -4\pi C \left[\frac{-1}{4} e^{-2r/a_o} (a_o^3 + 2ra_o^2 + 2a_o r) \right]_0^{a_o} \quad (24)$$

$$q(a_o) = \pi C a_o^3 \left(\frac{5}{e^2} - 1 \right) \approx -0.32\pi C a_o^3 \quad (25)$$

Note that here e is the base of the natural logarithm, *not* the electron charge!

What is the field at $r = a_o$? If the charge distribution is spherically symmetric, we can apply Gauss' law: it looks just like a point charge of magnitude $q(a_o)$. Thus,

$$\vec{E} = \frac{kq}{a_o^2} \hat{r} = \pi k C a_o \left(\frac{5}{e^2} - 1 \right) \hat{r} \quad (26)$$

Finally, we need the normalization constant C . We can find this by noting that the charge distribution integrated over all space must give the net charge for one electron, viz., $-e$. Rather than performing the integration over $r : 0 \rightarrow a_o$, we perform it over $r : 0 \rightarrow \infty$ and note $\lim_{r \rightarrow \infty} e^{-r/a_o} = 0$.

$$q_{\text{tot}} = -e = -4\pi C \int_0^{\infty} r^2 e^{-2r/a_o} dr = -4\pi C \left[\frac{-1}{4} e^{-2r/a_o} (a_o^3 + 2ra_o^2 + 2a_o r) \right]_0^{\infty} \quad (27)$$

$$-e = -4\pi C \left(-\frac{1}{4} a_o^3 \right) = \pi C a_o^3 \quad (28)$$

$$\implies C = \frac{e}{\pi a_o^3} \quad (29)$$

Note that here e is the electron charge ... and thus about 32% of the electron's total charge is contained within a radius a_o in this model based on our earlier result.

4. A plastic spherical shell has inner radius a and outer radius b . Electric charge is uniformly distributed over the region $a < r < b$, where r is the distance from the center of the spherical shell. The amount of charge is ρ coulombs per cubic meter. Find the electric field for regions $r \leq a$, $a \leq r \leq b$, and $r \geq b$.

This one is pure Gauss' law. Since the charge distribution is spherical, draw spherical surfaces centered on the center of the charge distribution. By symmetry, \vec{E} is constant on those surfaces, and the electric flux over the whole sphere is $E A = q_{\text{encl}} / \epsilon_o$.

First, how about the region $r \leq a$? Here, no charge is enclosed by a sphere of radius r , so $\vec{E} = 0$. Second, how about the region where $r \geq b$? Here we contain all of the charge. Given a constant density, we just need to know the volume of charge. That must simply be the volume of a sphere of radius b minus the volume of a sphere of radius a . Thus, the total charge for the whole shell is

$$q_{\text{tot}} = \frac{4}{3} \pi \rho (b^3 - a^3) \quad (30)$$

Since for $r \geq b$ we enclose all of this charge with a spherical surface, which has an area $4\pi r^2$

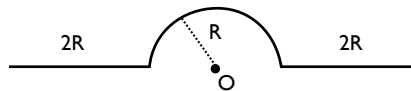
$$EA = 4\pi r^2 E = \frac{q_{\text{tot}}}{\epsilon_0} = \frac{4\pi\rho (b^3 - a^3)}{3\epsilon_0} \quad r \geq b \quad (31)$$

$$E = \frac{\rho (b^3 - a^3)}{3\epsilon_0 r^2} \quad r \geq b \quad (32)$$

Finally, what about for $a \leq r \leq b$? Here we don't enclose *all* of the charge, but just an amount equal to the charge density times the difference in volume of a sphere of radius r and a sphere of radius a . Basically: replace b with r above, and we have all three regions:

$$E = \begin{cases} 0 & r < a \\ \frac{\rho(r^3 - a^3)}{3\epsilon_0 r^2} & a \leq r \leq b \\ \frac{\rho(b^3 - a^3)}{3\epsilon_0 r^2} & r \geq b \end{cases} \quad (33)$$

5. *Serway & Jewett* A wire having uniform linear charge density λ is bent into the shape shown below. Find the electric potential at O .



Superposition is the name of the game! This is the same as the potential at the origin of a semicircle plus twice the potential of a line of charge $2r$ long at a distance r . Since potential is a *scalar*, we don't even need to worry about direction!

First, the semicircle. We already found the field in the first question, we could just find V from that. Let's do it from scratch, making use of the figure from problem 1. We have our charge dq at radius r , and we know its potential dV :

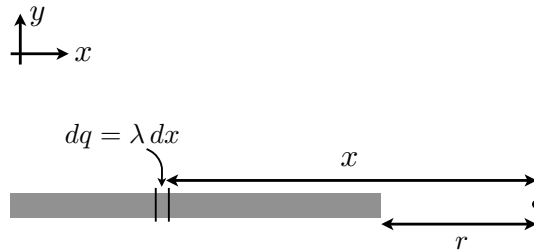
$$dV_{\text{semicircle}} = \frac{k dq}{r} = \frac{k\lambda r d\theta}{r} = k\lambda d\theta \quad (34)$$

Integrating this over the ring,

$$V_{\text{semicircle}} = \int_0^\pi k\lambda d\theta = k\lambda\pi \quad (35)$$

What about the lines? Each one gives the same contribution! Since potential is a scalar – no direction – they don't cancel, they simply add together. We then need the potential due to a line charge at a distance r along the axis. Set that up like this:

The charge of each little bit of the wire dx is $dq = \lambda dx$, and it is at a distance x . It gives a contribution to the



potential

$$dV_{\text{line}} = \frac{k dq}{x} = \frac{k \lambda dx}{x} \quad (36)$$

Here we want to integrate over the length of the rod, from $x=r$ to $x=3r$ (since each rod is $2r$ long):

$$V_{\text{line}} = \int_r^{3r} \frac{k \lambda dx}{x} = k \lambda \ln x \Big|_r^{3r} = k \lambda \ln 3 \quad (37)$$

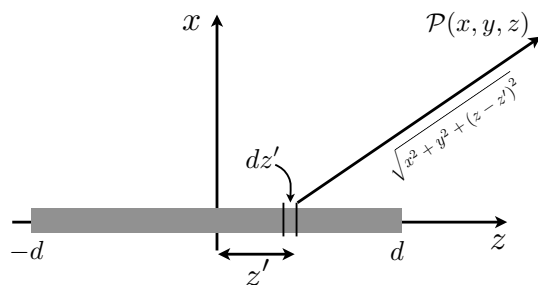
The total potential at the origin is then that of the semicircle plus two contributions from line charges:

$$V_{\text{tot}} = V_{\text{semicircle}} + 2V_{\text{line}} = k \lambda (\pi + 2 \ln 3) \quad (38)$$

6. *Purcell 2.10* A thin rod extends along the z axis from $z = -d$ to $z = d$. The rod carries a charge uniformly distributed along its length with a linear charge density λ . By integrating over this charge distribution calculate the potential at point P_1 on the z axis with coordinates $(0, 0, 2d)$. By another integration, find the potential at the point P_2 on the x axis and locate this point to make the potential equal to the potential at P_1 .

We're going to need it later anyway, so we may as well solve for the potential due to a thin rod of charge at an arbitrary point (x, y, z) . This will allow us to solve both parts of this problem at once, and we'll need the general solution next week anyway ... if we can solve two problems at once with just a little more math, we always do!

Set the general problem up like this:



We'll use z' to denote the position on the rod, our integration variable, and z to denote the position of interest \mathcal{P} . The distance from a small bit of charge dq to the point of interest $\mathcal{P}(x, y, z)$ is $\sqrt{x^2 + y^2 + (z - z')^2}$, and the potential due to this small bit is just

$$dV = \frac{k\lambda dz'}{\sqrt{x^2 + y^2 + (z - z')^2}} \quad (39)$$

The potential can be found at \mathcal{P} by integrating over the whole rod:

$$\begin{aligned} V &= k\lambda \int_{-d}^d \frac{dz'}{\sqrt{x^2 + y^2 + (z - z')^2}} = -k\lambda \ln \left[z - z' + \sqrt{x^2 + y^2 + (z - z')^2} \right] \Big|_{-d}^d \\ &= k\lambda \ln \left[\frac{z + d + \sqrt{x^2 + y^2 + (z + d)^2}}{z - d + \sqrt{x^2 + y^2 + (z - d)^2}} \right] \end{aligned} \quad (40)$$

Just for fun, let's get the components of the electric field too, using $\vec{E} = -\vec{\nabla}V$. Take the derivative before plugging in the limits of integration, it is somewhat easier.

$$\begin{aligned} E_x &= -\frac{dV}{dx} = \frac{-k\lambda x}{(z - z') \sqrt{x^2 + y^2 + (z - z')^2} + x^2 + y^2 + (z - z')^2} \Big|_{z'=-d}^{z'=d} \\ E_x &= \frac{-k\lambda x}{(z + d) \sqrt{x^2 + y^2 + (z + d)^2} + x^2 + y^2 + (z + d)^2} + \frac{k\lambda x}{(z - d) \sqrt{x^2 + y^2 + (z - d)^2} + x^2 + y^2 + (z - d)^2} \\ E_y &= -\frac{dV}{dy} = \frac{-k\lambda y}{(z + d) \sqrt{x^2 + y^2 + (z + d)^2} + x^2 + y^2 + (z + d)^2} + \frac{k\lambda y}{(z - d) \sqrt{x^2 + y^2 + (z - d)^2} + x^2 + y^2 + (z - d)^2} \\ E_z &= -\frac{dV}{dz} = \frac{k\lambda \left(\sqrt{x^2 + y^2 + (z - z')^2} + z - z' \right)}{(z - z') \sqrt{x^2 + y^2 + (z - z')^2} + x^2 + y^2 + (z - z')^2} \Big|_{z'=-d}^{z'=d} \\ E_z &= -\frac{k\lambda \left(\sqrt{x^2 + y^2 + (z + d)^2} + z + d \right)}{(z + d) \sqrt{x^2 + y^2 + (z + d)^2} + x^2 + y^2 + (z + d)^2} + \frac{k\lambda \left(\sqrt{x^2 + y^2 + (z - d)^2} + z - d \right)}{(z - d) \sqrt{x^2 + y^2 + (z - d)^2} + x^2 + y^2 + (z - d)^2} \end{aligned}$$

What a mess! Let's check a couple of high-symmetry points, such as $(x, 0, 0)$ $(0, 0, z)$ to see if it makes sense:

$$\begin{aligned} E_x(x, 0, 0) &= \frac{-k\lambda x}{x^2 + d^2 + d\sqrt{x^2 + d^2}} + \frac{k\lambda x}{x^2 + d^2 - d\sqrt{x^2 + d^2}} = \frac{2k\lambda d}{x\sqrt{x^2 + d^2}} \\ E_y(x, 0, 0) &= 0 \\ E_z(x, 0, 0) &= 0 \end{aligned}$$

$$E_x(0, 0, z) = 0$$

$$E_y(0, 0, z) = 0$$

$$E_z(0, 0, z) = \frac{2k\lambda}{z-d} - \frac{2k\lambda}{z+d}$$

Both of these check out with the standard solutions. Anyway: back to our regularly-scheduled problem! First, we want the potential at $\mathcal{P} = (0, 0, 2d)$:

$$V(\mathcal{P}) = V(0, 0, 2d) = k\lambda \ln \left[\frac{2d + d + 0}{2d - d + 0} \right] = k\lambda \ln 3 \quad (41)$$

Now we want the potential on the x axis, or $\mathcal{P}' = (x, 0, 0)$:

$$V(\mathcal{P}') = V(x, 0, 0) = k\lambda \ln \left[\frac{d + \sqrt{x^2 + d^2}}{-d + \sqrt{x^2 + d^2}} \right] \quad (42)$$

We want to find the points on the x axis $\pm x$ for which $V(\mathcal{P}) = V(\mathcal{P}')$, thus

$$k\lambda \ln \left[\frac{d + \sqrt{x^2 + d^2}}{-d + \sqrt{x^2 + d^2}} \right] = k\lambda \ln 3$$

Simplifying,

$$\begin{aligned} 3 &= \frac{d + \sqrt{x^2 + d^2}}{-d + \sqrt{x^2 + d^2}} \\ d + \sqrt{x^2 + d^2} &= -3d + 3\sqrt{x^2 + d^2} \\ 4d &= 2\sqrt{x^2 + d^2} \\ 4d^2 &= x^2 + d^2 \\ x &= \pm d\sqrt{3} \end{aligned} \quad (43)$$

7. *Purcell 1.28* Three protons and three electrons are to be placed at the vertices of a regular octahedron of edge length a . We want to find the potential energy of the system, or the work required to assemble it starting with the particles infinitely far apart. There are essentially two different arrangements possible. What is the energy of each? Symbolic answer, please.

Using the principle of superposition, we know that the potential energy of a system of charges is just the sum of the potential energies for all the unique pairs of charges. The problem is then reduced to figuring out how many different possible pairings of charges there are, and what the energy of each pairing is. The potential energy for a single pair of charges, both of magnitude q , separated by a distance d is just:

$$PE_{\text{pair}} = \frac{k_e q^2}{d}$$

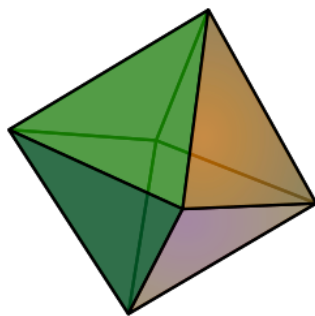


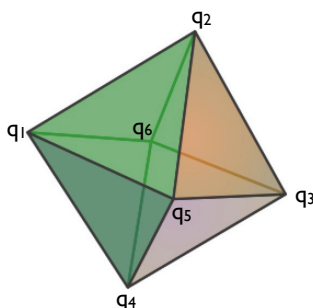
Figure 1: An octahedron. It has eight faces and six vertices.

First, we need figure out how many pairs there are for charges arranged on the vertices of an octahedron, and for each pair, how far apart the charges are. Once we've done that, we need to figure out the two different arrangements of charges and run the numbers.

How many unique pairs of charges are there? There are not so many that we couldn't just list them by brute force – which we will do anyway to calculate the energy – but we can also calculate how many there are. In both distinct configurations, we have 6 charges, and we want to choose all possible groups of 2 charges that are not repetitions. So far as potential energy is concerned, the pair (2, 1) is the same as (1, 2). Pairings like this are known as combinations, as opposed to *permutations* where (1, 2) and (2, 1) are *not* the same. Calculating the number of possible combinations is done like this:ⁱ

$$\text{ways of choosing pairs from six charges} = \binom{6}{2} = {}^6C_2 = \frac{6!}{2!(6-2)!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 15$$

We can verify this by simply enumerating all the possible pairings. Label the charges at each vertex in some fashion, such as this:

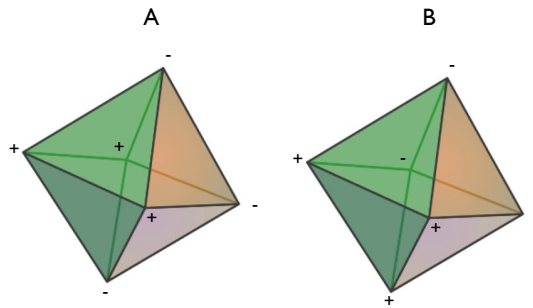


We have six charges at six vertices, and thus ${}^6C_2 = \frac{6!}{2!4!} = 15$ unique pairings of charges. Namely,

ⁱA nice discussion of combinations and permutations is here: <http://www.themathpage.com/aPreCalc/permutations-combinations.htm>

$$\begin{aligned}
& q_1 q_2, q_1 q_3, q_1 q_4, q_1 q_5, q_1 q_6 \\
& q_2 q_3, q_2 q_4, q_2 q_5, q_2 q_6 \\
& q_3 q_4, q_3 q_5, q_3 q_6 \\
& q_4 q_5, q_4 q_6 \\
& q_5 q_6
\end{aligned}$$

Here all the q_i have the same magnitude, the labels are just to keep things straight. At a given vertex, all four nearest-neighbor vertices are at distance a , while the single “next-nearest neighbor” is at a distance $a\sqrt{2}$. This means that there are three *pairs* charges which are separated by a distance $a\sqrt{2}$, and the other twelve pairings are at a distance a . We have highlighted the $a\sqrt{2}$ pairings above. How can we find two different arrangements? Since there are an odd number of next-nearest neighbor pairings, the first suspicion is that the difference between the two arrangements will be in next-nearest neighbor pairings. If you experiment for a while, the two different arrangements are these:



Now we need only add up the potential energies of all possible pairs of charges. All the nearest-neighbor pairs will have the same energy, viz.,

$$|U_{nn}| = \frac{kq^2}{a} \quad (44)$$

All the next-nearest neighbor pairs will have

$$|U_{nnp}| = \frac{kq^2}{a\sqrt{2}} \quad (45)$$

For the first arrangement we have 12 nearest-neighbor pairs: eight of them are $+ -$ pairings, and four of them are $++$ or $--$ pairs. We have three next-nearest neighbor pairs, two $++$ or $--$, and one $+ -$. Thus, the total energy must be

$$U_A = 8 \left[\frac{-kq^2}{a} \right] + 4 \left[\frac{kq^2}{a} \right] + 2 \left[\frac{kq^2}{a\sqrt{2}} \right] + 1 \left[\frac{-kq^2}{a\sqrt{2}} \right] = \frac{kq^2}{a} \left[\frac{1}{\sqrt{2}} - 4 \right] = \left[\frac{1}{\sqrt{2}} - 4 \right] |U_{nn}| \approx -3.29 |U_{nn}| \quad (46)$$

For the second arrangement, of the 12 nearest-neighbor pairs we have six $+ -$ pairs and six $++$ or $--$ pairs, and

thus the total energy of nearest-neighbor pairs will be zero. We are left with only the next-nearest neighbor terms, and for this arrangement, all three are $+ -$ pairs. Thus,

$$U_B = -3 \frac{kq^2}{a\sqrt{2}} = \frac{3}{\sqrt{2}} |U_{nn}| \approx -2.12 |U_{nn}| \quad (47)$$

Thus, $U_A < U_B$, and the first lattice is more stable, owing to its lower nearest-neighbor energy. Though the second lattice has a smaller next-nearest neighbor energy, there are fewer next-nearest neighbor pairs, and their energy is smaller than the nearest neighbor pairs. Usually, minimizing the nearest-neighbor energy gives the most stable crystal, simply because the potential is decreasing with distance.

8. *Purcell 2.22* An interstellar dust grain, roughly spherical with a radius of 3×10^{-7} m, has acquired a negative charge such that its electric potential is -0.15 Volts.

- (a) How many extra electrons has it picked up?
- (b) What is the strength of the electric field at its surface?

If it is spherical, Gauss' law tells us that we may treat it as a point charge (so long as we are outside the dust grain, anyway). The excess charge must therefore be equivalent to a point charge which at a distance 3×10^{-7} m creates a potential of -0.15 Volts. In terms of the number of electrons n , the net charge is $-ne$. Thus,

$$\begin{aligned} -0.15 \text{ V} &= \frac{-kne}{3 \times 10^{-7} \text{ m}} \\ n &\approx 31 \text{ electrons} \end{aligned} \quad (48)$$

Here we rounded to the nearest integer for n . The same point charge would produce an electric field at a distance of 3×10^{-7} m of

$$E = \frac{-kne}{(3 \times 10^{-7} \text{ m})^2} \approx 5 \times 10^5 \text{ V/m} \quad (49)$$

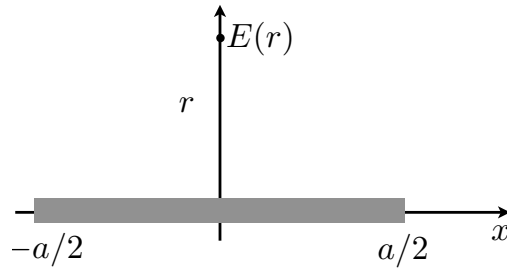
9. Find the electric field at a height z above the center of a square sheet (side a) carrying a uniform surface charge σ . Check your result for the limiting cases $a \rightarrow \infty$ and $z \gg a$.

We can build a plate of side a out of many rods of length a and infinitesimal thickness. From a previous problem above, we already have the electric field due to a line charge. In this case, we want the field at a point along an axis perpendicular to the rod, equidistant from both ends:

From our previous solution, noting $Q = \lambda a$, we have

$$E(r) = \frac{kQ}{r\sqrt{r^2 + a^2/4}} \quad (50)$$

For the present problem, we want to slice a square plate up into many thin rods. Let the rods run along the x axis, with infinitesimal thickness dy . Given the surface charge σ , each carries a charge $dQ = \sigma a dy$. Our plate will thus lie in the xy plane, with the origin at the center of the plate, meaning $x : -a/2 \rightarrow a/2$ and $y : -a/2 \rightarrow a/2$. We



seek the electric field at a distance z above the plate. By symmetry, the x and y components must vanish, and we need only worry about the z component. For any given infinitesimal rod, we know the electric field point from the center of the rod toward the field point. The distance from the center of a rod at a distance y from the origin to the field point $(0, 0, z)$ is

$$r = \sqrt{z^2 + y^2} \quad (51)$$

The electric field makes an angle θ with respect to the z axis, where

$$\cos \theta = \frac{z}{r} = \frac{z}{\sqrt{z^2 + y^2}} \quad (52)$$

The field contribution in the z direction from any infinitesimal rod is then given by the product of Eq. 50 and Eq. 52, using Eq. 51:

$$dE_z = \frac{kz dQ}{r^2 \sqrt{r^2 + a^2/4}} = \frac{k\sigma a z dy}{(z^2 + y^2) \sqrt{z^2 + y^2 + a^2/4}} \quad (53)$$

We need only integrate this over the width of the plate, $y : -a/2 \rightarrow a/2$.

$$E_z = \int_{-a/2}^{a/2} \frac{k\sigma a z dy}{(z^2 + y^2) \sqrt{z^2 + y^2 + a^2/4}} = 2 \int_0^{a/2} \frac{k\sigma a z dy}{(z^2 + y^2) \sqrt{z^2 + y^2 + a^2/4}} \quad (54)$$

For the last step, we recognized that the integral has the same value over $y : -a/2 \rightarrow 0$ and $y : 0 \rightarrow a/2$, and it is easier to do half the integral and double the result (since the limit at $y=0$ is an easy substitution). The integral is known to Wolfram:

$$E_z = 2 \int_0^{a/2} \frac{k\sigma a z dy}{(z^2 + y^2) \sqrt{z^2 + y^2 + a^2/4}} = 2k\sigma a z \left[\frac{2}{a z} \tan^{-1} \left(\frac{a y}{z \sqrt{a^2 + 4y^2 + 4z^2}} \right) \right] \Big|_0^{a/2}$$

$$E_z = 4k\sigma \tan^{-1} \left[\frac{a^2}{2z \sqrt{2a^2 + 4z^2}} \right] \quad (55)$$

We can readily find the limits $z \gg a$ and $a \rightarrow \infty$. For $z \gg a$, the argument of the arctangent function tends toward

zero. In this case, the argument can be approximated by

$$\frac{a^2}{2z\sqrt{2a^2 + 4z^2}} = \frac{a^2}{4z^2\sqrt{1 + \frac{a^2}{2z^2}}} \approx \frac{a^2}{4z^2} \quad (z \gg a) \quad (56)$$

Noting then that for small x , $\tan^{-1}(x) \approx x$,

$$E_z = 4k\sigma \tan^{-1} \left[\frac{a^2}{2z\sqrt{2a^2 + 4z^2}} \right] \approx \frac{4k\sigma a^2}{4z^2} = \frac{k\sigma a^2}{z^2} = \frac{kQ}{z^2} \quad (z \gg a) \quad (57)$$

Thus, we recover the form for a point charge when we are very far from the plate compared to its lateral dimensions. In the limit $a \rightarrow \infty$, the argument of the arctangent tends toward infinity. Noting, $\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$,

$$E_z = 4k\sigma \tan^{-1} \left[\frac{a^2}{2z\sqrt{2a^2 + 4z^2}} \right] \approx 4\pi k\sigma \quad (a \rightarrow \infty) \quad (58)$$

We thus recover the field due to an infinite sheet of charge, as expected.

One point remains: in the solutions manual, Griffiths calculates the field by another method, and finds the solution

$$E_z = 8k\sigma \left[\tan^{-1} \left(\sqrt{1 + \frac{a^2}{2z^2}} \right) - \frac{\pi}{4} \right] \quad (59)$$

You can show the two results to be equivalent using the rather obscure identity

$$\tan^{-1} \left(\frac{2u}{u^2 - 1} \right) = 2 \tan^{-1} \left(\frac{1}{u} \right) \pm n\pi \quad (n \in \mathbb{Z}) \quad (60)$$

Try a substitution $u^2 = 1 + \frac{a^2}{2z^2}$ in our expression and you should be able to show the equivalence of the two results. Given the freedom to add $\pm n\pi$ ($n \in \mathbb{Z}$) to an arctangent function, you should keep in mind the limiting cases for large a or z above to reproduce Griffiths expression.

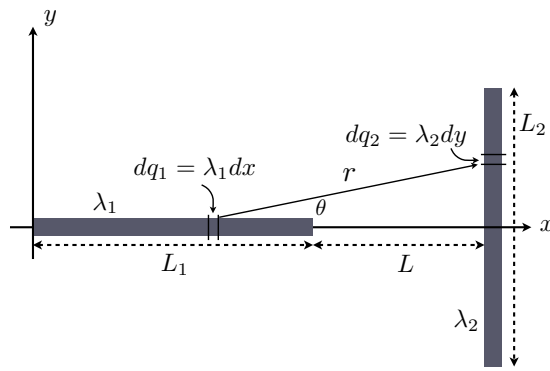
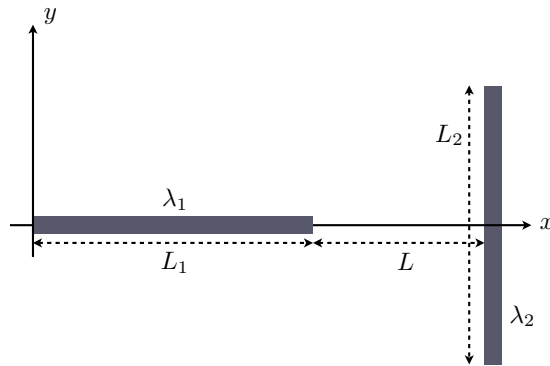
10. Two thin rigid rods lie along the x axis at right angles to one another, as shown below. Both rods are uniformly charged. Rod 1 has a length L_1 and a charge per unit length λ_1 . Rod 2 has a length L_2 and a charge per unit length λ_2 . The distance between the right end of rod 1 and rod 2 is L .

(a) Give an exact expression for the electrical force between the two rods, i.e. the force that one rod exerts on the other. If you get really stuck on the integral, you should always feel free to consult an integral table or try:

<http://integrals.wolfram.com>

(b) Show that in the limit $L \gg L_1$ and $L \gg L_2$ your expression for the force between the rods reduces to the Coulomb force between two point charges. What are the magnitudes Q_1 and Q_2 of the point charges?

This isn't as bad as it seems. We need to slice up each rod into little bits of charge, and find the force between a bit of charge in one rod and a bit of charge in the other. Then we can integrate over both rods to get the total force. First, set up the problem:



By symmetry, the net force in the y direction must vanish, so we need only worry about the x component of the force. The x component of the force between dq_1 and dq_2 , based on the sketch above, is

$$dF_x = \frac{k dq_1 dq_2}{r^2} \cos \theta = \frac{k \lambda_1 \lambda_2 dx dy}{(L + L_1 - x)^2 + y^2} \left[\frac{L + L_1 - x}{\sqrt{(L + L_1 - x)^2 + y^2}} \right] = \frac{k \lambda_1 \lambda_2 (L + L_1 - x) dx dy}{[(L + L_1 - x)^2 + y^2]^{3/2}} \quad (61)$$

The total force in the x direction is then found by integrating x over the first rod and y over the second:

$$\begin{aligned}
F_x &= \int_{-L_2/2}^{L_2/2} dy \int_0^{L_1} dx \frac{k\lambda_1\lambda_2 (L + L_1 - x)}{[(L + L_1 - x)^2 + y^2]^{3/2}} = \int_{-L_2/2}^{L_2/2} dy \left[\frac{k\lambda_1\lambda_2}{\sqrt{(L + L_1 - x)^2 + y^2}} \right]_0^{L_1} \\
&= \int_{-L_2/2}^{L_2/2} dy \left[\frac{k\lambda_1\lambda_2}{\sqrt{L^2 + y^2}} - \frac{k\lambda_1\lambda_2}{\sqrt{(L + L_1)^2 + y^2}} \right] \\
&= k\lambda_1\lambda_2 \left[\ln \left(y + \sqrt{y^2 + L^2} \right) - k\lambda_1\lambda_2 \ln \left(y + \sqrt{y^2 + (L_1 + L)^2} \right) \right]_{-L_2/2}^{L_2/2} \\
&= k\lambda_1\lambda_2 \left[\ln \left(L_2/2 + \sqrt{L_2^2/4 + L^2} \right) - \ln \left(-L_2/2 + \sqrt{L_2^2/4 + L^2} \right) \right] \\
&\quad - k\lambda_1\lambda_2 \left[\ln \left(L_2/2 + \sqrt{L_2^2/4 + (L_1 + L)^2} \right) + \ln \left(-L_2/2 + \sqrt{L_2^2/4 + (L_1 + L)^2} \right) \right] \\
&= k\lambda_1\lambda_2 \ln \left[\frac{\left(L_2/2 + \sqrt{L_2^2/4 + L^2} \right) \left(-L_2/2 + \sqrt{L_2^2/4 + (L_1 + L)^2} \right)}{\left(-L_2/2 + \sqrt{L_2^2/4 + L^2} \right) \left(L_2/2 + \sqrt{L_2^2/4 + (L_1 + L)^2} \right)} \right] \\
F_x &= k\lambda_1\lambda_2 \ln \left[\frac{\left(L_2 + \sqrt{L_2^2 + 4L^2} \right) \left(-L_2 + \sqrt{L_2^2 + 4(L_1 + L)^2} \right)}{\left(-L_2 + \sqrt{L_2^2 + 4L^2} \right) \left(L_2 + \sqrt{L_2^2 + 4(L_1 + L)^2} \right)} \right]
\end{aligned} \tag{62}$$

Now we need to find the limiting expression for $L \gg L_1, L_2$. It is a bit subtle. First, let's re-write what we have to make a sensible grouping of terms:

$$F_x = k\lambda_1\lambda_2 \ln \left[\left(\frac{L_2 + \sqrt{L_2^2 + 4L^2}}{-L_2 + \sqrt{L_2^2 + 4L^2}} \right) \left(\frac{-L_2 + \sqrt{L_2^2 + 4(L_1 + L)^2}}{L_2 + \sqrt{L_2^2 + 4(L_1 + L)^2}} \right) \right] \tag{63}$$

Let's tackle the first term in parentheses inside the logarithm first. If $L_1, L_2 \ll L$, then $L_2/L \approx 0$.

$$\frac{L_2 + \sqrt{L_2^2 + 4L^2}}{-L_2 + \sqrt{L_2^2 + 4L^2}} = \frac{L_2 + 2L\sqrt{L_2^2/4L^2 + 1}}{-L_2 + 2L\sqrt{L_2^2/4L^2 + 1}} \approx \frac{2L + L_2}{2L - L_2} = \frac{1 + \frac{L_2}{2L}}{1 - \frac{L_2}{2L}} \tag{64}$$

Now tackle the second term in parentheses inside the logarithm. Note also that if $L_1, L_2 \ll L$, then $L_1/L \approx 0$.

$$\frac{-L_2 + \sqrt{L_2^2 + 4(L_1 + L)^2}}{L_2 + \sqrt{L_2^2 + 4(L_1 + L)^2}} = \frac{-L_2 + 2L\sqrt{L_2^2/4L^2 + (L_1/L + 1)^2}}{L_2 + 2L\sqrt{L_2^2/4L^2 + (L_1/L + 1)^2}} = \frac{-L_2 + 2L\sqrt{1 + 2L_1/L + L_1^2/L^2 + L_2^2/4L^2}}{L_2 + 2L\sqrt{1 + 2L_1/L + L_1^2/L^2 + L_2^2/4L^2}} \tag{65}$$

If we are assuming $L_1, L_2 \ll L$, then we may neglect terms which are second order in L_1/L or L_2/L :

$$\frac{-L_2 + \sqrt{L_2^2 + 4(L_1 + L)^2}}{L_2 + \sqrt{L_2^2 + 4(L_1 + L)^2}} \approx \frac{-L_2 + 2L\sqrt{1 + 2L_1/L}}{L_2 + 2L\sqrt{1 + 2L_1/L}} \quad (66)$$

We can use a binomial expansion, viz., $(1 + x)^n \approx 1 + nx$ for $x \ll 1$:

$$\frac{-L_2 + \sqrt{L_2^2 + 4(L_1 + L)^2}}{L_2 + \sqrt{L_2^2 + 4(L_1 + L)^2}} \approx \frac{-L_2 + 2L(1 + L_1/L)}{L_2 + 2L(1 + L_1/L)} = \frac{2L + 2L_1 - L_2}{2L + 2L_1 + L_2} = \frac{1 + \frac{L_1}{L} - \frac{L_2}{2L}}{1 + \frac{L_1}{L} + \frac{L_2}{2L}} \quad (67)$$

Now we can multiply our two approximations, Eq. 64 and Eq. 67 to approximate the entire argument to the logarithm in Eq. 63

$$\left(\frac{L_2 + \sqrt{L_2^2 + 4L^2}}{-L_2 + \sqrt{L_2^2 + 4L^2}} \right) \left(\frac{-L_2 + \sqrt{L_2^2 + 4(L_1 + L)^2}}{L_2 + \sqrt{L_2^2 + 4(L_1 + L)^2}} \right) \approx \left[\frac{1 + \frac{L_2}{2L}}{1 - \frac{L_2}{2L}} \right] \left[\frac{1 + \frac{L_1}{L} - \frac{L_2}{2L}}{1 + \frac{L_1}{L} + \frac{L_2}{2L}} \right] \quad (68)$$

If we regroup these terms ...

$$\left[\frac{1 + \frac{L_2}{2L}}{1 + \frac{L_1}{L} + \frac{L_2}{2L}} \right] \left[\frac{1 - \frac{L_2}{2L}}{1 + \frac{L_1}{L} - \frac{L_2}{2L}} \right]^{-1} = \left[1 - \frac{\frac{L_1}{L}}{1 + \frac{L_1}{L} + \frac{L_2}{2L}} \right] \left[1 - \frac{\frac{L_1}{L}}{1 + \frac{L_1}{L} - \frac{L_2}{2L}} \right]^{-1} \equiv \frac{1 - \delta}{1 - \gamma} \quad (69)$$

Now if we put this back into our full expression for the force, Eq. 63, and note $\ln(1 + x) \approx x$ for $x \ll 1$,

$$F_x = k\lambda_1\lambda_2 \ln \left(\frac{1 - \delta}{1 - \gamma} \right) \approx k\lambda_1\lambda_2 (\gamma - \delta) = k\lambda_1\lambda_2 \left[\frac{\frac{L_1}{L}}{1 + \frac{L_1}{L} - \frac{L_2}{2L}} - \frac{\frac{L_1}{L}}{1 + \frac{L_1}{L} + \frac{L_2}{2L}} \right] \quad (70)$$

Simplifying again:

$$\frac{\frac{L_1}{L}}{1 + \frac{L_1}{L} - \frac{L_2}{2L}} - \frac{\frac{L_1}{L}}{1 + \frac{L_1}{L} + \frac{L_2}{2L}} = \frac{\frac{L_1}{L} + \frac{L_1^2}{L^2} + \frac{L_1L_2}{2L^2} - \frac{L_1}{L} - \frac{L_1^2}{L^2} + \frac{L_1L_2}{2L^2}}{\left(1 + \frac{L_1}{L}\right)^2 - \frac{L_2^2}{4L^2}} = \frac{L_1L_2}{L^2 \left(1 + \frac{2L_1}{L} + \frac{L_1^2}{L^2}\right) - \frac{L_2^2}{4}} \quad (71)$$

Now neglecting any fractions with L in the denominator, and remembering $L \gg L_2$,

$$\frac{L_1L_2}{L^2 \left(1 + \frac{2L_1}{L} + \frac{L_1^2}{L^2}\right) - \frac{L_2^2}{4}} \approx \frac{L_1L_2}{L^2 - \frac{L_2^2}{4}} \approx \frac{L_1L_2}{L^2} \quad (72)$$

Bringing us to

$$F_x \approx k\lambda_1\lambda_2 \frac{L_1L_2}{L^2} \quad (73)$$

If we identify $Q_1 = \lambda_1L_1$ and $Q_2 = \lambda_2L_2$ with the charge on the rods, we have (finally!) recovered Coulomb's law:

$$F_x \approx \frac{kQ_1Q_2}{L^2} \quad (74)$$