UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 126 / LeClair   Fall 2009

Problem Set 3: Potential, Capacitors

Instructions:

1. Answer all questions below.
2. Some problems have different due dates!
3. You may collaborate, but everyone must turn in their own work

The following problems are due Wed 9 September 2009

1. Consider the electric field of two protons (each with charge e) placed a distance b apart. According to Eq. 2.45 in Griffiths (which we derived in class), the potential energy of a system of charges is given by integrating the electric field over all space. In the present case, this ought to be given by something like Eq. 2.47 in Griffiths:

$$\begin{aligned} U &= \frac{1}{2}\epsilon_0 \int E^2\,d\tau = \frac{1}{2}\epsilon_0 \int \left(\vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2\right)^2\,d\tau \\ &= \frac{1}{2}\epsilon_0 \int \vec{\mathbf{E}}_1^2\,d\tau + \frac{1}{2}\epsilon_0 \int \vec{\mathbf{E}}_2^2\,d\tau + \epsilon_0 \int \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2\,d\tau \end{aligned}$$

where $\vec{\mathbf{E_1}}$ is the field of one particle alone, and $\vec{\mathbf{E_2}}$ is that of the other (and of course $d\tau$ is a differential element of volume). The first of the three integrals might be called the "electrical self-energy" of one proton; an intrinsic property of one particle, it depends on the proton's size and structure. We have always disregarded it in treating the potential energy of a system of charges, on the assumption that it remains constant. The same goes for the second integral.

The third integral involves the distance between the charges. It is not hard to evaluate if you set it up in spherical coordinates, with the origin on one of the charges and the other on the polar axis, integrating over r first. Thus, by direct calculation, you can show that the third integral has the value of ke^2/b, which we already know to be the work required to bring the two protons in from an infinite distance to positions a distance b apart. This will establish the correctness of Eq. 2.45 for this case, and by invoking superposition, you can argue that it must then give the energy required to assemble any system of charges.

2. Concentric spherical shells of radius a and b, with $b > a$, carry charge Q and $-Q$, respectively, each charge uniformly distributed. Find the energy stored in the electric field of this system.

3. Last week, in problem 6 you calculated the potential at two points near a charged rod of length $2d$, lying on the z axis from $z=-d$ to $z=d$. The two points at which you calculated the potential happen to lie on an ellipse which has the ends of the rod as its foci, as you can readily verify by comparing the sums of the distances from the two points to the ends of the rods. This suggests that the whole ellipse might be an equipotential (surface of constant potential).

(a) Test that conjecture by calculating the potential at the point $(3d/2, 0, d)$ which lies on the same ellipse.

(b) Indeed it is true, though there is no obvious reason why it should be, that the equipotential surfaces of this system are a family of confocal prolate spheroids. See if you can prove that. You will have to derive a formula for the potential at a general point $(x, 0, z)$ in the xz plane. Then show that, if x and z are related by the equation,

$$\frac{x^2}{a^2 - d^2} + \frac{z^2}{a^2} = 1$$

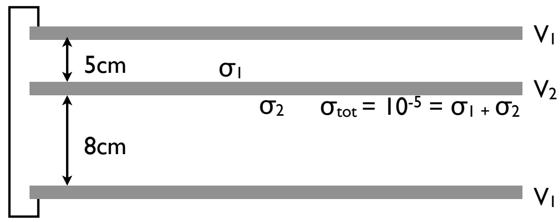
which is the equation for an ellipse with foci at $z = \pm d$, the potential will depend only on the parameter a, not on x or z.

4. Consider a charge distribution which has constant density ρ everywhere inside a cube of edge b and is zero everywhere outside that cube. Letting the electric potential V be zero at infinite distance from the cube of charge, denote by V_o the potential at the center of the cube and V_1 the potential at a corner of the cube. Determine the ratio V_o/V_1. The answer can be found with very little calculation by combining a dimensional argument with superposition. (Think about the potential at the center of a cube with the same charge density and twice the edge length.)

The following problems are due Fri 11 September 2009

5. Three conducting plates are placed parallel to one another as shown below. The outer plates are connected by a wire. The inner plate is isolated and carries a charge amounting to 10^{-5} C per square meter of plate. In what proportion must this charge divide itself into a surface charge σ_1 on one face of the inner plate and a surface charge σ_2 on the other side of the same plate?

Bonus: (+50% on this question) In addition, consider a situation where the middle plate is allowed to move up and down. More precisely, let D be the separation between the top and bottom plate, and let d be the separation between the top plate and the middle plate. Find the energy stored in the field as a function of d. For what value of d is this energy a maximum?



6. If you worked out problem 3, you should be able to derive from that result the capacitance C of an isolated conductor of prolate spheroid shape, viz.,

$$C = \frac{2a\epsilon}{\ln\left(\frac{1+\epsilon}{1-\epsilon}\right)} \quad \text{where} \quad \epsilon = \sqrt{1 - \frac{b^2}{a^2}}$$

where the spheroid has length $2a$ and diameter $2b$.

(a) Derive the expression for C above.
(b) Verify it reduces to the expression for a hollow sphere if $b = a$.
(c) Imagine the spheroid is a charged water drop. If this drop is deformed at constant volume and constant charge Q from a sphere to a prolate spheroid, will the energy stored in the electric field increase or decrease? (The volume of a prolate spheroid is $\frac{4}{3}\pi ab^2$.)

7. A capacitor consists of two concentric spherical shells. Call the inner shell, of radius a, conductor 1, and the outer shell, of radius b, conductor 2. For this two conductor system, find C_{11}, C_{22} and C_{12}. Recall that the C_{ij} relate to the potential and charge of each conductor:

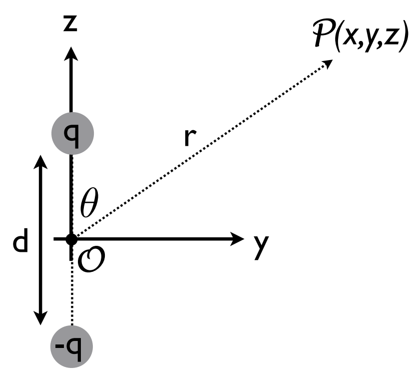
$$Q_i = \sum_i C_{ij} V_j \qquad \text{and} \qquad C_{ij} = C_{ji}$$

8. *Dipoles.* Consider a dipole consisting of two points charges q and $-q$, separated by a distance d, as shown below
(a) Show that for $r \gg d$ the potential produced by the dipole is

$$V(r,\theta) = \frac{qd\cos\theta}{r^2}$$

(b) Show that for $r \gg d$ the electric field produced by the dipole is

$$\vec{\mathbf{E}}(r,\theta) = \frac{qd}{r^3}\left(2\hat{\mathbf{r}}\,\cos\theta + \hat{\boldsymbol{\theta}}\,\sin\theta\right)$$



(c) What is the dipole moment $\vec{\mathbf{p}}$? Give your answer in terms of the Cartesian unit vectors.

(d) Show that the potential and field expressions can be rewritten in the following manner:

$$V(\vec{\mathbf{r}}) = \frac{\vec{\mathbf{p}} \cdot \hat{\mathbf{r}}}{r^2}$$
$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{(3\vec{\mathbf{p}} \cdot \vec{\mathbf{r}})\ \hat{\mathbf{r}} - \vec{\mathbf{p}}}{r^3}$$

Here $\vec{\mathbf{p}} = q\vec{\mathbf{d}}$, where $\vec{\mathbf{d}}$ is a vector connecting one charge to the other.