UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 126 / LeClair — Fall 2009

Problem Set 3: Solutions

1. *Purcell 1.35.* Consider the electric field of two protons (each with charge e) placed a distance b apart. According to Eq. 2.45 in Griffiths (which we derived in class), the potential energy of a system of charges is given by integrating the electric field over all space. In the present case, this ought to be given by something like Eq. 2.47 in Griffiths:

$$\begin{aligned} U &= \frac{1}{2}\epsilon_o \int E^2\, d\tau = \frac{1}{2}\epsilon_o \int \left(\vec{E}_1 + \vec{E}_2\right)^2\, d\tau \\ &= \frac{1}{2}\epsilon_o \int \vec{E}_1^2\, d\tau + \frac{1}{2}\epsilon_o \int \vec{E}_2^2\, d\tau + \epsilon_o \int \vec{E}_1 \cdot \vec{E}_2\, d\tau \end{aligned}$$

where $\vec{E}_1$ is the field of one particle alone, and $\vec{E}_2$ is that of the other (and of course $d\tau$ is a differential element of volume). The first of the three integrals might be called the "electrical self-energy" of one proton; an intrinsic property of one particle, it depends on the proton's size and structure. We have always disregarded it in treating the potential energy of a system of charges, on the assumption that it remains constant. The same goes for the second integral.

The third integral involves the distance between the charges. It is not hard to evaluate if you set it up in spherical coordinates, with the origin on one of the charges and the other on the polar axis, integrating over r first. Thus, by direct calculation, you can show that the third integral has the value of ke^2/b, which we already know to be the work required to bring the two protons in from an infinite distance to positions a distance b apart. This will establish the correctness of Eq. 2.45 for this case, and by invoking superposition, you can argue that it must then give the energy required to assemble any system of charges.

This problem is mainly a matter of getting the geometry right. Let the two protons be situated on the z axis, with the origin centered on the lower proton. Define a vector $\vec{b}$ pointing from the lower proton to the upper, i.e.,

$$\vec{b} = b\hat{z}$$

We first wish to determine the electric field due to each charge at an arbitrary point $\mathcal{P}$. Let $\vec{r}$ point from the lower proton (at the origin) to $\mathcal{P}$, which produces a field $\vec{E}_1$. Similarly, let $\vec{r}\,'$ point from the upper proton to $\mathcal{P}$, which produces a field $\vec{E}_2$. Overall, the situation looks like this:

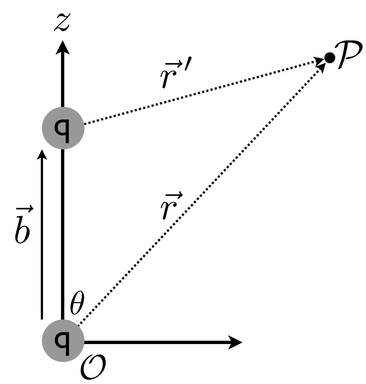


Figure 1: *Two protons separated by a distance* b.

We can already write down the electric field from both charges:

$$\vec{E}_1 = \frac{ke}{|\vec{r}|^2}\hat{r} = \frac{ke}{|\vec{r}|^3}\vec{r} \tag{1}$$

$$\vec{E}_2 = \frac{ke}{|\vec{r}\,'|^2}\hat{r}' = \frac{ke}{|\vec{r}\,'|^3}\vec{r}\,' \tag{2}$$

Here we used the definition $\hat{r} = \vec{r}/|\vec{r}|$. Since the upper and lower charges differ in position only by the constant vector $\vec{b}$, we know $\vec{r}\,' = \vec{r} - \vec{b}$. Thus,

$$\vec{E}_2 = \frac{ke}{|\vec{r}\,'|^3}\left(\vec{r} - \vec{b}\right) \tag{3}$$

$$\vec{E}_1 \cdot \vec{E}_2 = \frac{k^2e^2}{|\vec{r}|^3|\vec{r}\,'|^3}\vec{r} \cdot \left(\vec{r} - \vec{b}\right) = \frac{k^2e^2}{|\vec{r}|^3|\vec{r}\,'|^3}\left(|\vec{r}|^2 - \vec{r} \cdot \vec{b}\right) \tag{4}$$

Since we have aligned the pair of charges along the z axis, the remaining dot product is simple: $\vec{r} \cdot \vec{b} = |\vec{r}||\vec{b}|\cos\theta$. We can use the law of cosines to eliminate the last $\vec{r}\,'$ from our expression:

$$|\vec{r}\,'|^2 = |\vec{r}|^2 + |\vec{b}|^2 - 2|\vec{r}||\vec{b}|\cos\theta \tag{5}$$

Thus,

$$\vec{E}_1 \cdot \vec{E}_2 = \frac{k^2e^2\left(|\vec{r}|^2 - |\vec{r}||\vec{b}|\cos\theta\right)}{|\vec{r}|^3\left(|\vec{r}|^2 + |\vec{b}|^2 - 2|\vec{r}||\vec{b}|\cos\theta\right)^{3/2}} = \frac{k^2e^2\left(|\vec{r}| - |\vec{b}|\cos\theta\right)}{|\vec{r}|^2\left(|\vec{r}|^2 + |\vec{b}|^2 - 2|\vec{r}||\vec{b}|\cos\theta\right)^{3/2}} \tag{6}$$

Now everything is in term of a single radial distance $|\vec{r}|$ and the polar angle θ, and we can proceed with integration …but first we may as well drop the vector notation, now that we have only scalar quantities left. Let $r = |\vec{r}|$, $b = |\vec{b}|$ as usual. This leaves us with:

$$\epsilon_o \int \vec{E}_1 \cdot \vec{E}_2 \, d\tau = \epsilon_o \int \frac{k^2 e^2 (r - b\cos\theta)}{r^2 \left(r^2 + b^2 - 2rb\cos\theta\right)^{3/2}} \tag{7}$$

We have everything in terms of a radius and angle, so proceeding in spherical polar coordinates is the only thing that makes sense, really. This means that $d\tau = r^2 \sin\theta \, dr \, d\theta \, d\varphi$, with $r : 0 \to \infty$, $\theta : 0 \to \pi$, $\varphi : 0 \to 2\pi$. If we integrate over r first, we have a perfect differential[i]

$$\begin{aligned}
\epsilon_o \int \vec{E}_1 \cdot \vec{E}_2 \, d\tau &= \epsilon_o \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\infty} \frac{k^2 e^2 (r - b\cos\theta)}{r^2 \left(r^2 + b^2 - 2rb\cos\theta\right)^{3/2}} r^2 \sin\theta \, dr \\
&= \epsilon_o \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\infty} \frac{k^2 e^2 (r - b\cos\theta)\sin\theta}{\left(r^2 + b^2 - 2rb\cos\theta\right)^{3/2}} \, dr \\
&= \epsilon_o \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \left. \frac{-k^2 e^2 \sin\theta}{\sqrt{r^2 + b^2 - 2rb\cos\theta}} \right|_{r=0}^{r\to\infty} = \epsilon_o \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \frac{k^2 e^2 \sin\theta}{b} \\
&= \frac{\epsilon_o k^2 e^2}{b} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta = \frac{\epsilon_o k^2 e^2}{b} (4\pi) \qquad (\text{note } 4\pi\epsilon_o = 1/k)
\end{aligned} \tag{8}$$

$$\therefore \quad \epsilon_o \int \vec{E}_1 \cdot \vec{E}_2 \, d\tau = \frac{ke^2}{b} \tag{9}$$

2. *Purcell 1.30.* Concentric spherical shells of radius a and b, with $b > a$, carry charge Q and $-Q$, respectively, each charge uniformly distributed. Find the energy stored in the electric field of this system.

The last problem indicates one way that we may proceed: integrate the square of the electric field through all space. In this case, Gauss' law yields the field easily by considering spherical shells of radius r surrounding the charged spheres symmetrically:

$$\vec{E} = \begin{cases} 0 & b \leqslant r \\ \frac{-kQ}{r^2}\,\hat{\mathbf{r}} & a < r < b \\ 0 & r \leqslant a \end{cases} \tag{10}$$

The energy is now readily found. Given the spherical symmetry of the field, it only makes sense to integrate in spherical coordinates. Since the electric field is zero except for the region $a < r < b$, we need only integrate r over that range. Since the field does not depend on φ or θ, the integrals over those two angles just give us a factor 4π.

[i] Try substituting $u = r^2 - 2rb\cos\theta$ if you don't see it.

$$U = \frac{1}{2}\epsilon_o \int E^2\, d\tau = \frac{1}{8\pi k}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_a^b \frac{k^2Q^2}{r^4} r^2 \sin\theta\, dr = \frac{4\pi k^2 Q^2}{8\pi k}\int_a^b \frac{dr}{r^2} = \frac{kQ^2}{2}\left(\left.\frac{-1}{r}\right|_a^b\right)$$

$$\therefore \quad U = -\frac{kQ^2}{2}\left(\frac{1}{b} - \frac{1}{a}\right) \tag{11}$$

3. *Purcell 2.11.* Last week, in problem 6 you calculated the potential at two points near a charged rod of length $2d$, lying on the z axis from $z=-d$ to $z=d$. The two points at which you calculated the potential happen to lie on an ellipse which has the ends of the rod as its foci, as you can readily verify by comparing the sums of the distances from the two points to the ends of the rods. This suggests that the whole ellipse might be an equipotential (surface of constant potential).

(a) Test that conjecture by calculating the potential at the point $(3d/2, 0, d)$ which lies on the same ellipse.

(b) Indeed it is true, though there is no obvious reason why it should be, that the equipotential surfaces of this system are a family of confocal prolate spheroids. See if you can prove that. You will have to derive a formula for the potential at a general point $(x, 0, z)$ in the xz plane. Then show that, if x and z are related by the equation,

$$\frac{x^2}{a^2 - d^2} + \frac{z^2}{a^2} = 1$$

which is the equation for an ellipse with foci at $z=\pm d$, the potential will depend only on the parameter a, not on x or z.

Luckily, we calculated the potential at an arbitrary point (x, y, z) last week.

$$V(x,y,z) = k\lambda \ln\left[\frac{z + d + \sqrt{x^2+y^2+(z+d)^2}}{z - d + \sqrt{x^2+y^2+(z-d)^2}}\right] \tag{12}$$

Plugging in $(3d/2, 0, d)$, we find $V = k\lambda \ln 3$. How about showing that the equipotentials are prolate spheroids, or at least ellipses within the xz plane? Equipotentials are surfaces for which V is a constant, which means

$$k\lambda \ln\left[\frac{z + d + \sqrt{x^2+y^2+(z+d)^2}}{z - d + \sqrt{x^2+y^2+(z-d)^2}}\right] = C \quad \text{or} \quad \frac{z + d + \sqrt{x^2+y^2+(z+d)^2}}{z - d + \sqrt{x^2+y^2+(z-d)^2}} = C' \tag{13}$$

where $C' = e^{C/k\lambda}$. How can we show that this is the equation of an ellipse? There are a number of ways to go about this. The most general solution is nicely illustrated in the following paper, so we will not reproduce it here:

R.J. Rowley, "Finite line of charge," American Journal of Physics, Volume 74, pp. 1120-1125
`http://dx.doi.org/10.1119/1.2348889`

Another way is just to plug in the equation for an ellipse into our expression for V and show that it results in a constant potential. A general equation for an ellipse is given above; we can solve it for x^2 and substitute that into our expressions above.

$$\begin{aligned}
x^2 &= \left(a^2-d^2\right)\left(1-\frac{z^2}{a^2}\right) = a^2 - z^2 - d^2 + \frac{z^2d^2}{a^2} \\
\Longrightarrow \qquad \sqrt{x^2+(z+d)^2} &= \sqrt{a^2-z^2-d^2+\frac{z^2d^2}{a^2}+z^2+2zd+d^2}\sqrt{a^2+\frac{z^2d^2}{a^2}+2zd} \\
&= \frac{\sqrt{a^4+d^2z^2+2a^2zd}}{a} = \frac{a^2+zd}{a} \\
\sqrt{x^2+(z-d)^2} &= \frac{a^2-zd}{a}
\end{aligned} \tag{14}$$

Proceeding ...

$$\begin{aligned}
\frac{z+d+\sqrt{x^2+y^2+(z+d)^2}}{z-d+\sqrt{x^2+y^2+(z-d)^2}} &= \frac{z+d+\frac{a^2+zd}{a}}{z-d+\frac{a^2-zd}{a}} = \frac{az+ad+a^2+zd}{az-ad+a^2-zd} \\
&= \frac{z\left(a+d\right)+a\left(a+d\right)}{z\left(a-d\right)+a\left(a-d\right)} = \frac{a+d}{a-d} \\
\Longrightarrow \qquad V &= k\lambda \ln\left[\frac{a+d}{a-d}\right]
\end{aligned} \tag{15}$$

Thus, the potential is constant on ellipses whose semi-major and semi-minor axes depend only on the fixed quantities a and d. Showing that the equipotentials are *prolate spheriods* in three dimensions, rather than just ellipses in two dimensions, is just a matter of putting back the y^2 terms above, which changes nothing.

Plugging in a point at which we know the potential (e.g., $V(0,0,2d) = k\lambda \ln 3$) yields $a^2 = 4d^2$. The equipotential surface we dealt with in the previous problems is then

$$\frac{x^2}{3d^2}+\frac{z^2}{4d^2} = 1$$

Or, an ellipse with semi-major and semi-minor axes $2d$ and $d\sqrt{3}$, respectively.

4. *Purcell 2.30.* Consider a charge distribution which has constant density ρ everywhere inside a cube of edge b and is zero everywhere outside that cube. Letting the electric potential V be zero at infinite distance from the cube of charge, denote by V_o the potential at the center of the cube and V_1 the potential at a corner of the cube. Determine the ratio V_o/V_1. The answer can be found with very little calculation

by combining a dimensional argument with superposition. (Think about the potential at the center of a cube with the same charge density and twice the edge length.)

First, we can just brute force the solution, along with a couple of tricks involving *dimensionless arguments.* In order to find the potential at the center of the cube, V_o, we need to integrate the charge density over radial distance from the center over all space. Looks easy enough:

$$V_o = \int \frac{\rho}{r}\, d\tau \tag{16}$$

If we are at the center of the cube, with x, y, and z axes aligned along the cube axes, then we must integrate x, y, and z each over a range of $-b/2$ to $b/2$. We are basically forced to use rectangular coordinates here, meaning $d\tau = dx\, dy\, dz$ and $r^2 = x^2 + y^2 + z^2$. Thus,

$$V_o = \int_{-b/2}^{b/2}\int_{-b/2}^{b/2}\int_{-b/2}^{b/2} \frac{\rho\, dx\, dy\, dz}{\sqrt{x^2+y^2+z^2}} \tag{17}$$

This integral is analytically solvable, but there is no need. We can reduce it to a constant, independent of b if we introduce the dimensionless coordinates

$$\begin{aligned} x &= \overline{x}\left(\frac{b}{2}\right) & dx &= d\overline{x}\left(\frac{b}{2}\right) \\ y &= \overline{y}\left(\frac{b}{2}\right) & dy &= d\overline{y}\left(\frac{b}{2}\right) \\ z &= \overline{z}\left(\frac{b}{2}\right) & dz &= d\overline{z}\left(\frac{b}{2}\right) \end{aligned} \tag{18}$$

One idea here is that this integral (or one very similar to it) is going to come up again when we calculate V_1, the potential at the corner of the cube. By introducing dimensionless coordinates, we obtain the potential in a 'separated' form: the product of a function involving only our cube dimension b, and a function involving a general dimensionless integral. Performing the substitution:

$$V_o = \int_{-b/2}^{b/2}\int_{-b/2}^{b/2}\int_{-b/2}^{b/2} \frac{\rho\, dx\, dy\, dz}{\sqrt{x^2+y^2+z^2}} = \frac{1}{4}\rho b^2 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{d\overline{x}\, d\overline{y}\, d\overline{z}}{\sqrt{\overline{x}^2+\overline{y}^2+\overline{z}^2}} \tag{19}$$

We can simplify this further by noting that the argument of the integral is symmetric about $(0,0,0)$, and thus we can change the limits from $(-1,-1,-1) \rightarrow (1,1,1)$ to $(0,0,0) \rightarrow (1,1,1)$ and double the result.

$$V_o = \frac{1}{2}\rho b^2 \int_0^1\int_0^1\int_0^1 \frac{d\overline{x}\, d\overline{y}\, d\overline{z}}{\sqrt{\overline{x}^2+\overline{y}^2+\overline{z}^2}} \equiv \frac{1}{2}\rho b^2 C \tag{20}$$

As promised, the integral that remains does not depend on anything of interest to the problem, it is simply a number, and the important thing is *how the potential scales with b.* What is interesting about this is that it tells us that the potential of a cube scales as the square of its characteristic dimension, regardless of the actual size of the cube. Thus, a cube twice as large must have a potential at the center four times larger. More on that later ...

Next, we can set up an expression for the potential at the corner of the cube. In this case, we must integrate x, y, and z from one corner of the cube to the other:

$$V_1 = \int_0^b\int_0^b\int_0^b \frac{\rho\,dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}} \tag{21}$$

We can again introduce a set of dimensionless coordinates to make the integral independent of b:

$$\begin{aligned} x &= b\overline{x} \qquad dx = b\,d\overline{x} \\ y &= b\overline{y} \qquad dy = b\,d\overline{y} \\ z &= b\overline{z} \qquad dz = b\,d\overline{z} \end{aligned} \tag{22}$$

Performing the substitution,

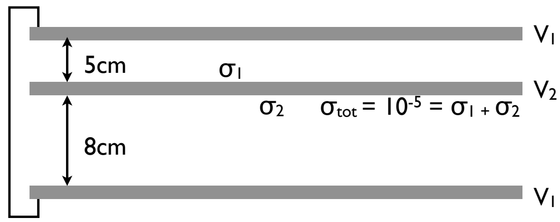
$$\begin{aligned} V_1 &= \rho b^2 \int_0^1\int_0^1\int_0^1 \frac{d\overline{x}\,d\overline{y}\,d\overline{z}}{\sqrt{\overline{x}^2+\overline{y}^2+\overline{z}^2}} = \rho b^2 C \\ \therefore \quad \frac{V_o}{V_1} &= 2 \end{aligned} \tag{23}$$

Thus, there is no need to actually calculate an integral: since both expressions end up with the same integral.

We could have been even a bit more sneaky using superposition, once we notice the scaling relationship $V_o \propto b^2$. Consider a cube of side $2b$. The potential at its center must be four times larger $4V_o$. However, we can think of building a cube of side $2b$ out of eight smaller cubes of side b. At the center of the larger cube, the corners of eight smaller cubes meet. Thus, by superposition, *the potential at the center of the larger cube must be eight times the potential at the corner of a smaller cube*, V_1. Putting these two bits together, it must be true that $4V_o = 8V_1$, or $V_o = 2V_1$.

5. *Purcell 3.8.* Three conducting plates are placed parallel to one another as shown below. The outer plates are connected by a wire. The inner plate is isolated and carries a charge amounting to 10^{-5} C per square meter of plate. In what proportion must this charge divide itself into a surface charge σ_1 on one face of the inner plate and a surface charge σ_2 on the other side of the same plate?

Bonus: (+50% on this question) In addition, consider a situation where the middle plate is allowed to move up and down. More precisely, let D be the separation between the top and bottom plate, and let d be the separation between the top plate and the middle plate. Find the energy stored in the field as a function of d. For what value of d is this energy a maximum?



Let's tackle the specific case first, and then work out the more general case. If we connect the two outer plates, they are effectively part of the same conductor, and thus they will be at the same potential. Call that V_1. Let the inner plate be at potential V_2. Conservation of charge dictates that the total surface charge on plate two is $\sigma = \sigma_1 + \sigma_2 = 10^{-5}\,\mathrm{C/m}$.

In the region between the upper plate and the middle plate, which we will call region I, the total electric field must be the sum of the field from the upper plate and that of the middle plate. The field from the middle plate in this region is that of an infinite plate with surface charge σ_1, $E = \sigma_1/2\epsilon_o$. The upper plate, under the influence of the top side of the middle plate, will have an induced charge $-\sigma_1$. It will contribute the same electric field in the region between the upper and middle plates,[ii] and in total we have

$$E_I = \sigma_1/2\epsilon_o + \sigma_1/2\epsilon_o = \sigma_1/\epsilon_o \tag{24}$$

The electric potential between the upper and middle plates must be $V_2 - V_1$, and it must be equivalent to integrating the electric field across the gap between the plates. Since the electric field is independent of distance, this is easy:

$$V_2 - V_1 = -\int_I \vec{E}_I \cdot d\vec{l} = El = E\,(5\,\mathrm{cm}) = (5\,\mathrm{cm})\,\sigma_1/\epsilon_o \tag{25}$$

Proceeding similarly in region II between the lower and middle plates, we find

$$V_2 - V_1 = (8\,\mathrm{cm})\,\sigma_2/\epsilon_o \tag{26}$$

Dividing the last two equations, we find

$$(5\,\mathrm{cm})\,\sigma_1/\epsilon_o = (8\,\mathrm{cm})\,\sigma_2/\epsilon_o \qquad \Longrightarrow \qquad \frac{\sigma_1}{\sigma_2} = \frac{8}{5} \tag{27}$$

[ii] The induced charge on the upper plate is negative, but the field is in the opposite direction.

Noting $\sigma = \sigma_1 + \sigma_2$, we find

$$\sigma_1 = \frac{8}{13}\sigma \qquad \sigma_2 = \frac{5}{13}\sigma \tag{28}$$

What about the more general case? Let the spacing between the upper and middle plates be d, and the spacing between the upper and lower plates be D (and thus the spacing between the lower and middle plates is $D-d$). Proceeding as above, we still have $E_I = \sigma_1/\epsilon_o$, and

$$V_2 - V_1 = -\int_I \vec{E}_I \cdot d\vec{l} = El = Ed = d\sigma_1/\epsilon_o \tag{29}$$

In the region between the lower and middle plates, we have $E_{II} = \sigma_2/\epsilon_o$, and

$$V_2 - V_1 = -\int_{II} \vec{E}_{II} \cdot d\vec{l} = El = E\,(D-d) = (D-d)\,\sigma_2/\epsilon_o \tag{30}$$

Thus,

$$\frac{\sigma_1}{\sigma_2} = \frac{D-d}{d} \tag{31}$$

Again noting $\sigma = \sigma_1 + \sigma_2$,

$$\sigma_1 = \left(\frac{D-d}{D}\right)\sigma \qquad \sigma_2 = \left(\frac{d}{D}\right)\sigma \tag{32}$$

We can find the energy stored by integrating the electric field squared over all space. Outside all of the plates, $\vec{E} = 0$. We can break up the integral over the region between the two plates into an integral over region I and an integral over region II. Since the electric field is constant in each region, the integrals simply reduce to the volume contained in the region between the two plates. Assume each plate as an area A.

$$\begin{aligned} U &= \frac{1}{2}\epsilon_o \int E^2\, d\tau = \frac{1}{2}\epsilon_o \int_I E_I^2\, d\tau + \frac{1}{2}\epsilon_o \int_{II} E_{II}^2\, d\tau = \frac{1}{2}\epsilon_o\,(\sigma_1/\epsilon_o)^2 \int_I d\tau + \frac{1}{2}\epsilon_o\,(\sigma_2/\epsilon_o)^2 \int_{II} d\tau \\ U &= \frac{\sigma_1^2}{2\epsilon_o} Ad + \frac{\sigma_2^2}{2\epsilon_o} A\,(D-d) \end{aligned} \tag{33}$$

Now we may substitute $\sigma_2 = \frac{d}{D}\sigma$ and $\sigma_1 = \left(\frac{D-d}{d}\right)\sigma$:

$$\begin{aligned} U &= \frac{\sigma_1^2}{2\epsilon_o} Ad + \frac{\sigma_2^2}{2\epsilon_o} A\,(D-d) = \frac{A}{2\epsilon_0}\left[d\left(\frac{D-d}{D}\sigma\right)^2 + \left(\frac{d}{D}\sigma\right)^2 (D-d)\right] \\ &= \frac{A\sigma^2}{2D^2\epsilon_0}\left[d\,(D-d)^2 + d^2\,(D-d)\right] = \frac{A\sigma^2}{2D^2\epsilon_0}\left(dD^2 - 2Dd^2 + d^3 + d^2D - d^3\right) \\ \therefore \quad U &= \frac{A\sigma^2}{2D^2\epsilon_0}\left(dD^2 - d^2D\right) = \frac{A\sigma^2}{2D\epsilon_0}\left(dD - d^2\right) \end{aligned} \tag{34}$$

We wish to optimize U with respect to d:

$$\frac{dU}{dd} = \frac{A\sigma^2}{2D\epsilon_0}(D-2d) = 0 \qquad \Longrightarrow \qquad d = \frac{D}{2} \tag{35}$$

We can verify this is a *maximum* potential energy by noting $d^2U/dd^2 < 0$ for all d. A nice result: maximal energy is stored for a symmetric placement of the middle plate, just what we might have expected. Another way to approach this problem would be to notice that this is really just two capacitors connected in series, which leads you to the same result.

6. If you worked out problem 3, you should be able to derive from that result the capacitance C of an isolated conductor of prolate spheroid shape, viz.,

$$C = \frac{2a\epsilon}{k\ln\left(\frac{1+\epsilon}{1-\epsilon}\right)} \quad \text{where} \quad \epsilon = \sqrt{1-\frac{b^2}{a^2}}$$

where the spheroid has length $2a$ and diameter $2b$.[iii]

(a) Derive the expression for C above.
(b) Verify it reduces to the expression for a hollow sphere if $b=a$.
(c) Imagine the spheroid is a charged water drop. If this drop is deformed at constant volume and constant charge Q from a sphere to a prolate spheroid, will the energy stored in the electric field increase or decrease? (The volume of a prolate spheroid is $\frac{4}{3}\pi ab^2$.)

Here we make use of our previous work on line charges. The idea is that the capacitance of a conducting prolate spheroid can be calculated by noting that the field due to charge a charge Q on the spheroid is the same as that due to charge Q spread uniformly along a line charge. Basically: if a line charge produces a surface of constant potential which is a prolate spheroid, then it is electrostatically indistinguishable from an *actual* conducting prolate spheroid with a charge. We already know the potential due to a line charge:

$$V(x,y,z) = k\lambda \ln\left[\frac{z+d+\sqrt{x^2+y^2+(z+d)^2}}{z-d+\sqrt{x^2+y^2+(z-d)^2}}\right] \tag{36}$$

Further, we already showed that the equipotential surfaces may be simply represented by

$$V = k\lambda\ln\left[\frac{a+d}{a-d}\right] \qquad \text{with} \qquad \frac{x^2}{a^2-d^2} + \frac{z^2}{a^2} = 1 \tag{37}$$

If we note that our line charge has a total charge $Q = 2d\lambda$, we have

$$V = \frac{kQ}{2d}\ln\left[\frac{a+d}{a-d}\right] \tag{38}$$

[iii] The original problem set was missing the k in the formula for capacitance above.

Capacitance is just charge per unit potential,

$$C = \frac{Q}{V} = \frac{2d}{k}\frac{1}{\ln\left[\frac{a+d}{a-d}\right]} = \frac{2d}{k}\frac{1}{\ln\left[\frac{1+d/a}{1-d/a}\right]} \tag{39}$$

We also can simplify the expression for eccentricity by noting that the axes our prolate spheroid are a and $b = \sqrt{a^2 - d^2}$, and thus

$$\epsilon = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{a^2 - d^2}{a^2}} = \frac{d}{a} \tag{40}$$

This also implies $2d = 2a\epsilon$. Thus,

$$C = \frac{2d}{k}\frac{1}{\ln\left[\frac{1+d/a}{1-d/a}\right]} = \frac{2a\epsilon}{k\ln\left[\frac{1+\epsilon}{1-\epsilon}\right]} \tag{41}$$

If we let the prolate spheriod become a sphere, then $b = a$, and $\epsilon \to 0$. Since $\ln(1+x) \approx x$ for small x,

$$\ln\left[\frac{1+\epsilon}{1-\epsilon}\right] = \ln[1+\epsilon] - \ln[1-\epsilon] \approx 2\epsilon \tag{42}$$

Thus, we recover the capacitance of a sphere

$$C = \frac{2a\epsilon}{k\ln\left[\frac{1+\epsilon}{1-\epsilon}\right]} \approx \frac{a}{k} = 4\pi\epsilon_o a \tag{43}$$

What about the energy stored in the electric field of a prolate spherical conductor? Under the condition of constant charge Q, we can write the stored energy as $U = Q^2/2C$, or

$$U_p = \frac{Q^2}{2C} = \frac{kQ^2}{4a\epsilon}\ln\left[\frac{1+\epsilon}{1-\epsilon}\right] \tag{44}$$

We are interested in seeing how this energy develops from a spherical conductor, where $\epsilon = 0$ and $a = b$. Therefore, if we expand the energy for small values of ϵ, we should find the energy of a spherical conductor plus higher-order terms giving the net excess or deficit of energy due to the deformation. Using our logarithm expansion in the expression for the energy of a prolate spheroid,

$$U_p \approx \frac{kQ^2}{4a\epsilon}(2\epsilon) = \frac{kQ^2}{2a} \tag{45}$$

This would be the energy of a spherical conductor of radius r ... but we don't *have* a sphere. We need to find the relationship between the major and minor axes of the spheroid, a and b, and the original radius of the sphere b. If a sphere is to be deformed at constant volume from a radius r to a prolate spheroid of axes a and b, then we require

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi a b^2 \qquad \Longrightarrow \qquad a = \frac{r^3}{b^2} \tag{46}$$

What does the deformation from a sphere to a prolate spheroid mean, physically? A prolate spheroid has a major axis a and a minor axis b. If we are to deform a sphere of radius r into a prolate spheroid, it means that we must increase the polar axis a and decrease the equatorial axis b, or $b < r < a$. Thus, under the condition of constant volume, while a increases as the deformation proceeds, b must decrease relative to the initial radius of the sphere r. Using the substitution above, we find

$$U_p \approx \frac{kQ^2b^2}{2r^3} = \frac{kQ^2}{2r}\left(\frac{b^2}{r^2}\right) \tag{47}$$

Our original spherical shell of radius r and charge Q has an energy of

$$U_s = \frac{kQ^2}{2r} \tag{48}$$

noting that for a spherical shell, $C = r/k$ and $U = Q^2/2C$. The energy difference due to the deformation is thus

$$\Delta U = U_p - U_s \approx \frac{kQ^2}{2r}\left(\frac{b^2}{r^2}\right) - \frac{kQ^2}{2r} = \frac{kQ^2}{2r}\left(\frac{b^2}{r^2} - 1\right) \tag{49}$$

Since the semi-minor axis b must shrink relative to r, and thus $b/r < 1$, the energy of the droplet *decreases* upon deformation: a charged prolate spheroidal water droplet is more stable than a perfectly spherical droplet. The relative change in energy is

$$\frac{\Delta U}{U_s} \approx \frac{b^2}{r^2} - 1 < 0 \tag{50}$$

We have not taken into account the increased surface energy of the prolate spheroidal droplet relative to the spherical droplet, which scales as $\sim \epsilon$. This increased surface energy favors the spherical droplet, and the actual equilibrium state will then be a function of the surface tension and size of the droplet. However, one can show that in the presence of an external electric field (as you would find in a raincloud), the prolate spheroidal shape is indeed the equilibrium state. See, for example,

> S.S. Abbi, and R. Chandra, Proc. Nat. Sci. India A, 1956, pg. 363
> `www.new.dli.ernet.in/rawdataupload/upload/.../20005acf_363.pdf`

7. A capacitor consists of two concentric spherical shells. Call the inner shell, of radius a, conductor 1, and the outer shell, of radius b, conductor 2. For this two conductor system, find C_{11}, C_{22} and C_{12}. Recall that the C_{ij} relate to the potential and charge of each conductor:

$$Q_i = \sum_i C_{ij} V_j \qquad \text{and} \qquad C_{ij} = C_{ji}$$

We should first remind ourselves what the coefficients of capacitance are. In general, for a system of many conductors, the coefficients of capacitance relate the charge Q_i and potential Q_j on a given conductor to the charge and potential of the other conductors, Q_j and V_j. Though it seems like a complex problem, we can build up a system of many conductors at arbitrary potentials with arbitrary charges by superposition.

Let's start with a system of just two conductors where one has a charge $+Q$ and the other has charge $-Q$. If conductor 1 has potential V_1 and conductor 2 has potential V_2, then we know $Q \propto V_1 - V_2$, which was the starting point for our definition of capacitance, viz., $Q = C(V_1 - V_2)$. The constant of proportionality between the potential *difference* of the two conductors and the charge one each is capacitance.

This simple analysis doesn't quite work if we have a larger number of conductors – we need all the potential differences between pairs of conductors and the charges on each conductor. Of course, potential is always relative to some reference point, and since we can only measure differences in potential, we can make that reference at any point we like. Rather than worry about many potential differences, we can simply define some convenient spot to be our "zero" of potential and quote the potential of each conductor relative to it. This special point is our *ground*, where we define $V = 0$. This eliminates the need for specifying many potential differences in favor of just specifying the potential on each conductor (rather than specifying $\binom{N}{2} = N(N-1)/2$ potential differences, we need only N potentials relative to ground). Put another way, this is just superposition. If we know the potential difference between conductors i and j and between conductors j and k, we also know the potential difference between i and k: $V_{ik} = V_{ij} - V_{jk}$. All we really need is one reference point, our ground, and the potentials of each conductor relative to it.

Let's say we have N conductors, with charges $Q_1, Q_2, \ldots Q_i \ldots Q_N$ and potentials $V_1, V_2, \ldots V_i \ldots V_N$. What we would like is a way to specify the Q's in terms of the V's. Clearly, a simple constant C will no longer do it. Mathematically, we could represent the V_i and Q_i by vectors, which means the 'constant' of proportionality we seek is really a *matrix* $\boldsymbol{C}$. The elements of the matrix $\boldsymbol{C}$, the C_{ij}, are our coefficients of capacitance, which relate the charge on one conductor to the charge and potential of all other conductors:

$$Q_i = \sum_{j=1}^{N} C_{ij} V_j \tag{51}$$

Or, in matrix form,

$$\begin{bmatrix} Q_1 \\ \vdots \\ Q_N \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1N} \\ C_{21} & C_{22} & C_{23} & \cdots & \vdots \\ C_{31} & C_{32} & C_{33} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{N1} & C_{N2} & C_{N3} & \ldots & C_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_N \end{bmatrix} \tag{52}$$

We would like a way to specify this matrix in terms of the geometry of the situation, without worrying about the particular details as much as possible, just as we did for capacitance. One important thing we already know: the matrix $\boldsymbol{C}$ must be symmetric about the diagonal, $C_{ij} = C_{ji}$, since the relevant potential differences are a property of *pairs* of conductors. Put more simple, pairing conductors i and j is the same thing as pairing conductors j and i, so it must be so. If conductor i is placed at potential V_o, and conductor j is grounded, then conductor i will induce some charge Q_o on conductor j. On the other hand, if conductor i is grounded and conductor j has the same potential V_o, then the same charge Q_o will be induced on conductor i. The situation is symmetric.[iv] What about the rest? This is where superposition really shows its power.

Before we consider all conductors different potentials, consider first the situation where all conductors are grounded except the first one (i.e., $V_i = 0, i \neq 1$). The charge on the first conductor is now easily specified for this first problem:

$$\begin{aligned} Q_1(1) &= C_{11}V_1 \\ Q_2(1) &= C_{21}V_1 \\ Q_3(1) &= C_{31}V_1 \\ &\cdots\cdots\cdots \\ Q_N(1) &= C_{N1}V_1 \end{aligned} \tag{53}$$

On the other hand, we could just as easily ground all but the *second* conductor, which leads us to a solution to this second problem:

$$\begin{aligned} Q_1(2) &= C_{12}V_2 \\ Q_2(2) &= C_{22}V_2 \\ Q_3(2) &= C_{32}V_2 \\ &\cdots\cdots\cdots \\ Q_N(2) &= C_{N2}V_2 \end{aligned} \tag{54}$$

If we continued along these lines, grounding all but one conductor i in turn and finding the charge on each conductor, superposition of all those N solutions gives us the solution to the general problem all N conductors at different potentials with different charges. The sum of our N solutions together gives the potential on each conductor in terms of the coefficients of capacitance and the potential of each conductor. For the first conductor, this yields:

[iv] This is not exactly a proof that $C_{ij} = C_{ji}$, but merely a strong indication that it must be true. The proof of this fact, known as *Green's reciprocity theorem*, is somewhat more detailed than we would like. A good reference is *Principles of Electrodynamics*, M. Schwartz, Ch. 2

$$Q_1 = C_{11}V_1 + C_{12}V_2 + \ldots + C_{1N}V_N = \sum_{j=1}^{N} C_{1j}V_j \tag{55}$$

Or, considering all conductors, we recover our matrix equation above:

$$Q_i = \sum_{j=1}^{N} C_{ij}V_j \tag{56}$$

The advantage of this method, along with the requirement $C_{ij} = C_{ji}$, is that once we determine the full set of C_{ij} for any given configuration of conductors, we can get all the potentials V_i from the charges Q_i, or vice versa. We still need the coefficients, however. How does one start to tackle such a problem? We have basically two types of coefficients: the "self" capacitance terms, C_{ii}, and the "induced" capacitance terms, C_{ij}. Using the superposition method above:

$$\begin{aligned} & C_{ii} \Rightarrow \text{charge on conductor } i \text{ with } V_i \text{ while other conductors are grounded} \\ \text{or} \quad & C_{ii} = \frac{Q_i}{V_i} \end{aligned} \tag{57}$$

$$\begin{aligned} & C_{ij} \Rightarrow \text{charge on grounded conductor } i \text{ with conductor } j \text{ at } V_j\text{, all others grounded} \\ \text{or} \quad & C_{ij} = \frac{Q_i}{V_j} = C_{ji} = \frac{Q_j}{V_i} \end{aligned} \tag{58}$$

Now, back to the problem at hand: in its most general guise, it is just a system of two conductors, in which case our general equations read

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{59}$$

or without the matrix notation,

$$\begin{aligned} Q_1 &= C_{11}V_1 + C_{12}V_2 \\ Q_2 &= C_{21}V_1 + C_{22}V_2 \\ C_{21} &= C_{12} \end{aligned} \tag{60}$$

Now it is just a matter of considering the different physical situations that the coefficients of capacitance represent.

Case C_{11}**:** First, we wish to determine C_{11}. This solution corresponds to conductor 1 at potential V_1 with charge Q_1, while conductor 2 is grounded ($V_2 = 0$) and possesses charge Q_2. We need to determine

V_1 in terms of Q_1 to find C_{11}.

Let the inner spherical conductor of radius a be conductor 1, and the outer conductor of radius b be conductor 2. Let the origin lie at the center of both spheres. If the inner sphere has charge Q_1, the outer conducting sphere will have an induced charge of $-Q_1$ on its inner surface – since it is grounded, it can pull as many charges from the earth as necessary to make this happen. What we need to determine is the potential of conductor 1. If we can find the electric field due to the charged spheres, we can find the potential as well.

Outside of both spheres, we have a net enclosed charge of $Q_1 + (-Q_1) = 0$, and the field must be zero by Gauss' law. This makes sense: if the outer conducting sphere is grounded, its induced charge will always act to perfectly compensate for any excess charge inside, and the field is *always* zero outside a grounded conducting object. Between the two spheres, the field also follows from Gauss' law. A sphere centered on the origin of radius r, $a < r < b$ encloses only the central sphere, so the enclosed charge is simply Q_1. Due to the spherical symmetry, the field in this region must then be the same as that of a point charge Q_1 at the origin. For radii smaller than a, we again have no enclosed charge, and the field is zero.

$$\begin{aligned}\vec{E} &= 0 \qquad r < a \\ \vec{E} &= \frac{kQ_1}{r^2}\hat{\mathbf{r}} \qquad a < r < b \\ \vec{E} &= 0 \qquad r > b\end{aligned} \tag{61}$$

The potential of conductor 1, the inner sphere of radius a, can then be found by integrating $\vec{E} \cdot d\vec{l}$ from a reference point of zero potential to the surface of conductor 1 at $r = a$. Typically, we integrate from ∞ down to $r = a$, since ∞ is our typical reference for $V = 0$. In this case, since conductor 2 is at zero potential, it is a perfectly valid reference, and we need only integrate from b to a.[v] The most convenient path for our integral is simply to walk along the radial direction, $d\vec{l} = \hat{\mathbf{r}}\,dr$.

$$\begin{aligned}V_1 = V(r=a) - V(r=b) &= -\int_b^a \vec{E} \cdot d\vec{l} = -\int_b^a \frac{kQ_1}{r^2}dr = kQ_1 \left[\frac{1}{r}\right]\Bigg|_b^a \\ &= kQ_1\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{kQ_1\,(b-a)}{ab}\end{aligned} \tag{62}$$

The coefficient C_{11} is now easily found:

$$C_{11} = \frac{Q_1}{V_1} = \frac{Q_1}{\frac{kQ_1(b-a)}{ab}} = \frac{ab}{k\,(b-a)} \tag{63}$$

[v] Or if you like, integrate from ∞ to a, but the field is zero from ∞ to b anyway …

Case C_{12}, C_{21}**:** Now we wish to determine either C_{12} or C_{21}, which must be equivalent. The first corresponds to the situation where conductor 2 has potential V_2 with conductor 1 grounded, and we wish to find the ratio of the charge on conductor 1 (Q_2) to the potential on conductor 1 (V_1).

Let the inner sphere of radius a be grounded, and thus $V_1 = 0$, while the outer sphere of radius b is at potential V_2. If the outer sphere has a charge Q_2, then a charge Q_1 must be induced on the inner sphere. Once again, we can find the electric field everywhere by Gauss' law, and it is the same as in the previous example: the field is non-zero only for radii $a<r<b$, where it is identical to a point charge of magnitude Q_1. We can again find the potential by integrating $\vec{E} \cdot d\vec{l}$ between the two spheres, from the . This time, the inner sphere has zero potential, and serves as our reference point .

$$
\begin{aligned}
V_2 = V(r=b) - V(r=a) &= -\int_a^b \vec{E} \cdot d\vec{l} = -\int_a^b \frac{kQ_1}{r^2} dr = kQ_1 \left[\frac{1}{r}\right]\Bigg|_a^b \\
&= kQ_1 \left(\frac{1}{b} - \frac{1}{a}\right) = \frac{-kQ_1 (b-a)}{ab}
\end{aligned} \tag{64}
$$

The coefficient C_{12} is now easily found:

$$
C_{12} = \frac{Q_1}{V_2} = \frac{Q_1}{\frac{-kQ_1(b-a)}{ab}} = \frac{-ab}{k(b-a)} = C_{21} \tag{65}
$$

Case C_{22}**:** This situation corresponds to the outer conductor 2 having potential V_2 and charge Q_2 while the inner conductor 1 is grounded, as in the previous case. We wish to find $C_{22} = Q_2/V_2$, but this time we will proceed somewhat differently.

If the outer sphere has charge Q_2 and the inner sphere Q_1, then outside of both spheres Gauss' law says that the field should be the same as a point charge of magnitude $Q_1 + Q_2$. Since we still require that the potential vanish at ∞, the potential of the second sphere is

$$
V_2 = V(r=b) - V(\infty) = -\int_\infty^b \vec{E} \cdot d\vec{l} = -\int_\infty^b \frac{k(Q_1+Q_2)}{r^2} dr = \frac{k(Q_1+Q_2)}{b} \tag{66}
$$

We wish to find the potential V_2 in terms of Q_2 only, meaning we must eliminate Q_1 in some way. Thankfully, from our previous calculation of C_{12}, we know what Q_1 is in terms of V_2:

$$
Q_1 = C_{12} V_2 = \frac{-abV_b}{k(b-a)} \tag{67}
$$

Substituting,

$$V_b = \frac{kQ_1}{b} + \frac{kQ_2}{b} = \frac{k}{b}\left(\frac{-abV_b}{k(b-a)}\right) + \frac{kQ_2}{b} = \frac{-aV_b}{b-a} + \frac{kQ_2}{b} \tag{68}$$

$$V_b\left(1 + \frac{a}{b-a}\right) = \frac{kQ_2}{b} \tag{69}$$

$$V_b\left(b + \frac{ab}{b-a}\right) = V_b\left(\frac{b^2}{b-a}\right) = kQ_2 \tag{70}$$

$$\implies \quad C_{22} = \frac{Q_2}{V_2} = \frac{b^2}{k(b-a)} \tag{71}$$

Thus, the desired coefficients of capacitance are

$$C_{11} = \frac{ab}{k(b-a)} \tag{72}$$

$$C_{12} = C_{21} = \frac{-ab}{k(b-a)} \tag{73}$$

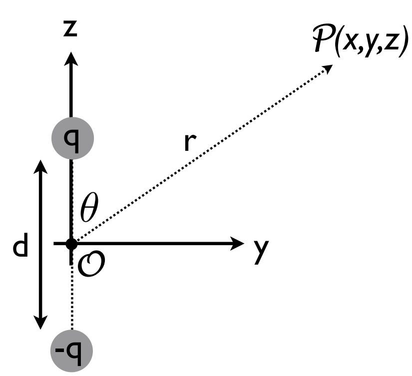
$$C_{22} = \frac{b^2}{k(b-a)} \tag{74}$$

Or, in matrix form:

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \frac{kb}{b-a}\begin{bmatrix} a & -a \\ -a & b \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{75}$$

Having solved the relevant sub-problems, we have now by superposition also solved the case of arbitrary potentials and charges on either conductor.

8. *Dipoles.* Consider a dipole consisting of two points charges q and $-q$, separated by a distance d, as shown below



(a) Show that for $r \gg d$ the potential produced by the dipole is

$$V(r,\theta) = \frac{kqd\cos\theta}{r^2}$$

(b) Show that for $r \gg d$ the electric field produced by the dipole is

$$\vec{E}(r,\theta) = \frac{kqd}{r^3}\left(2\hat{r}\,\cos\theta + \hat{\theta}\,\sin\theta\right)$$

(c) What is the dipole moment $\vec{p}$? Give your answer in terms of the Cartesian unit vectors.

(d) Show that the potential and field expressions can be rewritten in the following manner:

$$V(\vec{r}) = \frac{k\vec{p}\cdot\hat{r}}{r^2}$$
$$\vec{E}(\vec{r}) = k\left[\frac{(3\vec{p}\cdot\vec{r})\;\hat{r} - \vec{p}}{r^3}\right]$$

Here $\vec{p} = q\vec{d}$, where $\vec{d}$ is a vector connecting one charge to the other.

Figure 3 shows the dipole we wish to study, two charges q separated by d. We will choose the origin $\mathcal{O}$ to be precisely between the two charges, along the line connecting the charges, and we wish to calculate the field at an arbitrary point $\mathcal{P}(x,y,z) = \mathcal{P}(r)$ far from the dipole ($r \gg d$).[vi].

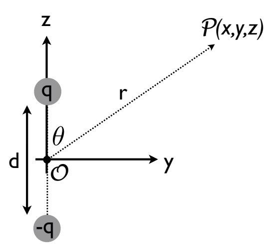


Figure 2: *An electric dipole consisting of two charges* q *separated by* d. *The origin* $\mathcal{O}$ *is chosen between the two charges, and we wish to calculate the field at an arbitrary point* $\mathcal{P}(x,y,z) = \mathcal{P}(r)$ *far from the dipole (*$r \gg d$*).*

We can readily write down the potential at the point $\mathcal{P}$ – it is just a superposition of the potential due to each of the charges alone:

$$V(x,y,z) = k_e\left[\frac{q}{\sqrt{\left(z-\frac{d}{2}\right)^2 + x^2 + y^2}} + \frac{-q}{\sqrt{\left(z+\frac{d}{2}\right)^2 + x^2 + y^2}}\right] \tag{76}$$

Since we are assuming $r \gg d$, we can simplify this a bit by noting that $x^2 + y^2 + z^2 = r^2 \gg d^2$ and collecting terms:

[vi]It is not much harder to solve without assuming $r \gg d$, but we don't really *need* the solution close to the dipole.

$$\frac{1}{\sqrt{\left(z-\frac{d}{2}\right)^2+x^2+y^2}}=\frac{1}{\sqrt{z^2-zd+\frac{d^2}{4}+x^2+y^2}}$$

$$=\frac{1}{\sqrt{x^2+y^2+z^2}}\left(\frac{1}{\sqrt{1-\frac{zd}{x^2+y^2+z^2}+\frac{d^2}{4x^2+4y^2+4z^2}}}\right)\approx\frac{1}{r}\left(\frac{1}{\sqrt{1-\frac{zd}{r}}}\right)$$

If we now use a binomial approximation, $(1+a)^n \approx na$ for $a \ll 1$, things look much nicer:

$$\frac{1}{\sqrt{\left(z-\frac{d}{2}\right)^2+x^2+y^2}}\approx=\frac{1}{r\sqrt{1-\frac{zd}{r^2}}}\approx\frac{1}{r}\left(1+\frac{zd}{2r^2}\right) \tag{77}$$

We can do this for both terms in the potential, and arrive at a fairly simple form for V:

$$V(r)=k_e\frac{z}{r^3}qd \tag{78}$$

Now we define a vector quantity known as the *electric dipole moment* as $\vec{p}=qd\,\hat{z}=q\vec{d}$, where $\vec{d}$ is just a vector going from one charge to the other. This is a single vector that categorizes the "strength" of our dipole. The dipole moment is a property of both charges, and it scales with the magnitude of the two charges, and their separation. If we also notice that z/r is nothing more than $\cos\theta$, the angle between $\vec{r}$ and the z axis. With this in hand,

$$V(r)=k_e\frac{|\vec{p}|\cos\theta}{r^2} \tag{79}$$

Finally, $|\vec{p}|\cos\theta$ is nothing more than a dot product of $\vec{p}$ and the radial unit vector $\hat{\mathbf{r}}$. This allows us to produce a nice, compact form of the electric potential due to a dipole which doesn't depend on any particular choice of coordinate system. Further, it encapsulates the orientation-dependence of the field more clearly:

$$V(r)=k_e\frac{\vec{p}\cdot\hat{\mathbf{r}}}{r^2}=k_e\frac{\vec{p}\cdot\vec{r}}{r^3} \tag{80}$$

This is a coordinate-free form for potential relatively far from an electric dipole.

How do we find the electric field resulting from this dipole? We take the gradient, in spherical coordinates. Let's try to do this part in a coordinate-free manner, so far as possible. First, the radial part. We'll need to use the chain rule:

$$E_r=-\frac{\partial V}{\partial r}=-k_e\frac{\partial}{\partial r}\left(\frac{\vec{p}\cdot\hat{\mathbf{r}}}{r^2}\right)=-\frac{k_e}{r^2}\left(\frac{\partial\left(\vec{p}\cdot\hat{\mathbf{r}}\right)}{\partial r}\right)-k_e\vec{p}\cdot\hat{\mathbf{r}}\left(\frac{\partial}{\partial r}\frac{1}{r^2}\right) \tag{81}$$

The angular part looks like this:[vii]

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\frac{k_e}{r}\frac{\partial}{\partial \theta}\left(\frac{\vec{p}\cdot\hat{\mathbf{r}}}{r^2}\right) = -\frac{k_e}{r^3}\left(\frac{\partial\left(\vec{p}\cdot\hat{\mathbf{r}}\right)}{\partial \theta}\right) - \frac{k_e}{r}\vec{p}\cdot\hat{\mathbf{r}}\left(\frac{\partial}{\partial \theta}\frac{1}{r^2}\right) \tag{82}$$

For the radial part, the first term vanishes, since $\vec{p}$ is independent of r and $\hat{\mathbf{r}}$ is a constant vector in the radial direction:

$$\frac{\partial\left(\vec{p}\cdot\hat{\mathbf{r}}\right)}{\partial r} = \frac{\partial \vec{p}}{\partial r}\cdot\hat{\mathbf{r}} + \vec{p}\cdot\frac{\partial \hat{\mathbf{r}}}{\partial r} = 0 \tag{83}$$

The second term for the radial part is easy, and the result is

$$E_r = \frac{2k_e\vec{p}\cdot\hat{\mathbf{r}}}{r^3} = \frac{2k_e|\vec{p}|\cos\theta}{r^3} \tag{84}$$

The angular part is a bit trickier. Let's handle the first term, which has the following derivative:

$$\frac{\partial\left(\vec{p}\cdot\hat{\mathbf{r}}\right)}{\partial \theta} = \vec{p}\cdot\frac{\partial \hat{\mathbf{r}}}{\partial \theta} + \left(\frac{\partial \vec{p}}{\partial \theta}\right)\cdot\hat{\mathbf{r}} \tag{85}$$

The second term above vanishes – since $\vec{p}$ points along a fixed direction, its derivative with respect to θ is zero. The first term does not: you may recall that the rate of change of the radial unit vector with respect to an angle is the angular unit vector, or $\partial\hat{\mathbf{r}}/\partial\theta = \hat{\theta}$. Thus,

$$\frac{\partial\left(\vec{p}\cdot\hat{\mathbf{r}}\right)}{\partial r} = \vec{p}\cdot\hat{\theta} \tag{86}$$

What we have left for the angular part is a term containing

$$k_e\vec{p}\cdot\hat{\mathbf{r}}\left(\frac{\partial}{\partial \theta}\frac{1}{r^2}\right) \tag{87}$$

This clearly vanishes, since r does not depend on θ. Thus,

$$E_\theta = \frac{k_e\vec{p}\cdot\hat{\theta}}{r^3} = -\frac{k_e|\vec{p}|\sin\theta}{r^3} \tag{88}$$

For the last step, we substituted $\vec{p} = |\vec{p}|\hat{\mathbf{z}}$ and noted $\hat{\mathbf{z}}\cdot\hat{\theta} = -\sin\theta$. In total, we have

$$\vec{E} = \frac{k_e}{r^3}\left(2\vec{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \frac{k_e}{r^3}\left(\vec{p}\cdot\hat{\theta}\right)\hat{\theta} = \frac{k|\vec{p}|}{r^3}\left[2\hat{\mathbf{r}}\cos\theta + \hat{\theta}\sin\theta\right] \tag{89}$$

We can now resolve the dipole moment along radial and angular directions. That simply means dotting $\vec{p}$ into each of those unit vectors:

$$\vec{p} = \left(\vec{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} + \left(\vec{p}\cdot\hat{\theta}\right)\hat{\theta} \tag{90}$$

[vii] Remember we are in spherical coordinates!

The form above looks very similar to this, except for the weighting the components. We can fix that ... if we notice the following trick:

$$\begin{aligned}(3\vec{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\vec{p} &= 3\,(\vec{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\left[(\vec{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}+\left(\vec{p}\cdot\hat{\theta}\right)\hat{\theta}\right]\\ &= 2\,(\vec{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\left(\vec{p}\cdot\hat{\theta}\right)\hat{\theta}\end{aligned} \tag{91}$$

And, we are done:

$$\therefore \qquad \vec{E}=\frac{k}{r^3}\left[(3\vec{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\vec{p}\right] \tag{92}$$

At the end of all this, we are left with a few important facts. First, the field strength from a dipole falls off as $1/r^3$, faster than for a point charge. Second, along the dipole axis, the field is parallel to the dipole moment $\vec{p}$, with magnitude $2|\vec{p}|/r^3$. Third, in the equatorial plane (the $x-y$ plane), the field points *antiparallel* to $\vec{p}$, and has the value $-\vec{p}/r^3$. These facts will be crucial for understanding the polarizability of dielectrics, and their ability to store electrical energy.

What do the field and potential look like? Below we plot some equipotentials (blue) and electric field lines (red) near a dipole. Note that the equipotentials and electric field lines are perpendicular everywhere, as they must be since $\vec{E}=-\vec{\nabla}V$.

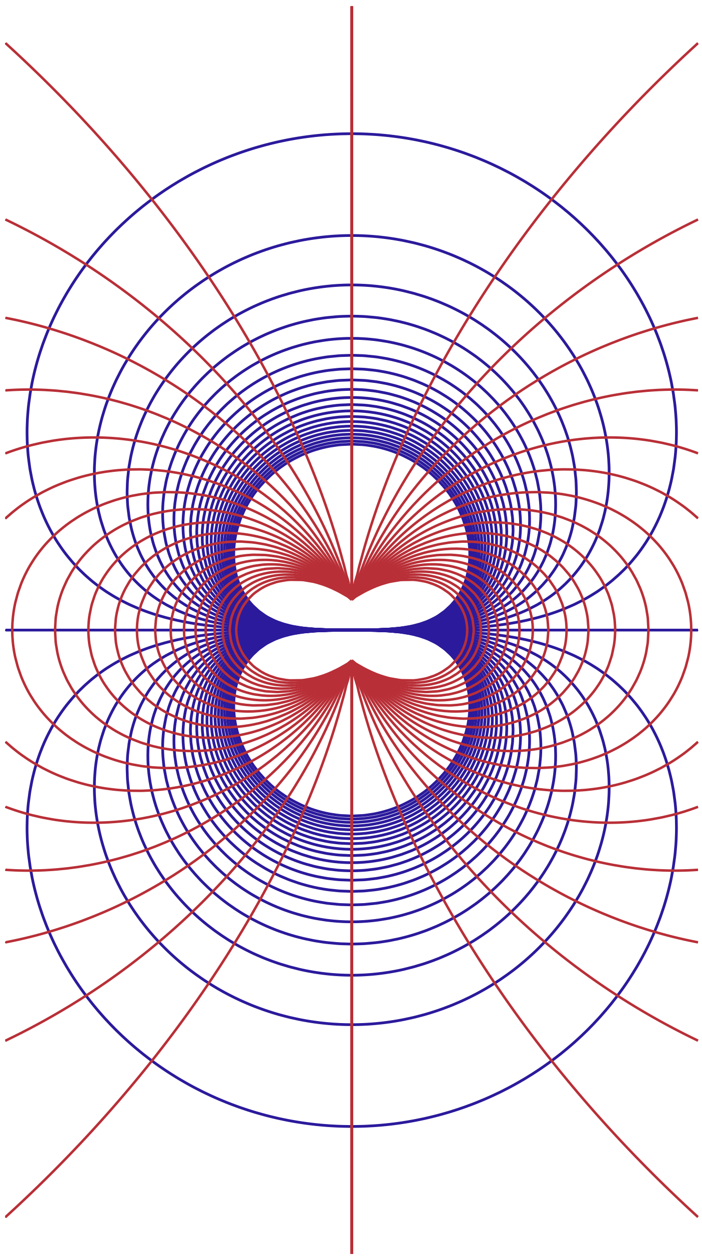


Figure 3: *Equipotentials (blue) and electric field lines (red) near a dipole*