UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 126 / LeClair Fall 2009

Problem Set 4: Solutions

1. Two resistors are connected in parallel, with values R_1 and R_2. A total current I_o divides somehow between them. Show that the condition $I_1 + I_2 = I_o$, together with the requirement of minimum power dissipation, leads to the same current values that we would calculate with normal circuit formulas. This illustrates a general variational principle that holds for direct current networks: the distribution of currents within the networks, for a given input current I_o, is always that which gives the least total power dissipation.

First, let's figure out the current in each resistance using the normal circuit formulas. Since the two resistors are in parallel, they will have the same potential difference across them, but in general different currents (unless $R_1 = R_2$, in which case the currents are the same). Let I_1 and I_2 be the currents in resistors R_1 and R_2, respectively, with the total current then given by conservation of charge, $I_o = I_1 + I_2$. Given the current through each resistor, we can readily calculate the voltage drop on each, which again must be the same for both resistors:

$$\Delta V_1 = I_1 R_1$$
$$\Delta V_2 = I_2 R_2$$
$$\Delta V_1 = \Delta V_2 \qquad \Longrightarrow \qquad I_1 R_1 = I_2 R_2$$

We can find the current in each resistor from the known total current I_o by noting that $I_o = I_1 + I_2$, and thus $I_2 = I_o - I_1$

$$I_1 R_1 = I_2 R_2$$
$$I_1 R_1 = (I_o - I_1) R_2$$
$$I_1 R_1 + I_1 R_2 = I_o R_2$$
$$\Longrightarrow \qquad I_1 = \frac{R_2}{R_1 + R_2} I_o = \left[\frac{1}{1 + \frac{R_1}{R_2}}\right] I_o$$

Thus, the fraction of the total current in first resistor depends on the ratio of the two resistors. The larger resistor 2 is, the more current that will flow through the first resistor – not shocking! Given the above expression for I_1, we can easily find I_2 from $I_2 = I_o - I_1$, which yields

$$I_2 = \frac{R_1}{R_1 + R_2} I_o = \left[\frac{1}{1 + \frac{R_2}{R_1}}\right] I_o$$

Our derivation of the currents in each resistor has so far only relied on conservation of energy (components in parallel have the same voltage) and conservation of charge ($I_o = I_1 + I_2$), we have not invoked any special "laws" about combining parallel resistors. In fact, that is what we have just derived!

Now for the requirement of minimum power dissipation. We want to find the distribution of currents that results in minimum power dissipation in the most general way, specifically *not* using the results of the previous portion of this problem. We will only assume that resistors R_1 and R_2 carry currents I_1 and I_2, respectively, and that these two currents add up to the total current, $I_o = I_1 + I_2$. In other words, we only assume conservation of charge to start with.

The total power dissipated is just the sum of the individual power dissipations in the two resistors:

$$\mathscr{P}_{\text{tot}} = \mathscr{P}_1 + \mathscr{P}_2 = I_1^2 R_1 + I_2^2 R_2 = I_1^2 R_1 + \left(I_o - I_1\right)^2 R_2 = I_1^2\left(R_1 + R_2\right) + I_o^2 R_2 - 2IR_2 I_1$$

For the last part, we invoked our conservation of charge condition ($I_o = I_1 + I_2$). What to do next? We have now the total power $\mathscr{P}_{\text{tot}}$ in both resistors as a function of the current in R_1. If we minimize the total power with respect to I_1, we will have found the value of I_1 which leads to the minimum power dissipation. Since I_2 is then fixed by the total current I once we know I_1, $I_2 = I_o - I_1$, this is sufficient to establish the values of *both* I_1 and I_2 that lead to minimum power dissipation. Of course, to find the minimum of $\mathscr{P}_{\text{tot}}$ for any value of I_1, we need to take a derivative[i] ...

$$\frac{d\mathscr{P}_{\text{tot}}}{dI_1} = 2I_1\left(R_1 + R_2\right) - 2I_o R_2 = 0$$

$$\Longrightarrow I_1 = \frac{R_2}{R_1 + R_2} I_o$$

Lo and behold, the minimum power dissipation occurs when the currents are distributed exactly as we expect for parallel resistors. At this point, you can easily find I_2 as well, given $I_2 = I_o - I_1$. The general rule is that current in a dc circuit distributes itself such that the total power dissipation is minimum, which we will not prove here.

Of course ... hold on a minute. We missed one small point: by finding $\frac{d\mathscr{P}_{\text{tot}}}{dI_1}$ and setting it to zero, we have certainly found an extreme value for $\mathscr{P}_{\text{tot}}$. We did not prove whether it is a maximum or a minimum however. This is exactly the sort of shenanigans we need to avoid to do proper physics, so we should apply the *second derivative test*.

[i]Keep in mind that the total current I_o is fixed, so $dI_o/dI_1 = 0$. And, yes we should technically be using partial derivatives here (differentiating with respect to I_1 while holding everything else constant), but since only I_1 varies that would be a bit pedantic. Plus, the ∂ symbols seem to scare people.

Recall briefly that after finding the extreme point of a function $f(x)$ via $df/dx|_{x=a}= 0$, one should calculate $d^2f/dx^2|_{x=a}$: if $d^2f/dx^2|_{x=a}< 0$, you have a maximum, if $d^2f/dx^2|_{x=a}> 0$ you have a minimum, and if $d^2f/dx^2|_{x=a}= 0$, the test basically wasted your time. Anyway:

$$\frac{d^2\mathscr{P}_{tot}}{dI_1^2} = 2\left(R_1 + R_2\right) > 0$$

Since resistances are always positive, we have in fact found a minimum of $\mathscr{P}_{tot}$. Crisis averted.

Don't let that lull you into complacency, however: you always need to apply the second derivative test to see what you've really found. At the very least, you should invoke the symmetry of the function to justify having found a minimum or maximum, and not just take derivatives and set them to zero all willy-nilly.

2. Show that if a battery of fixed internal voltage ΔV and internal resistance r is connected to a variable external resistance R the maximum power is delivered to the external resistor when $r = R$.

The circuit we are considering is just a series combination of the (ideal) internal voltage source ΔV, the internal resistance R_i, and the external resistance R. Since all the elements are in series, the current is the same in each, which we will call I. Applying conservation of energy,

$$\Delta V - IR - IR_i = 0 \qquad \Longrightarrow \qquad I = \frac{\Delta V}{R + R_i}$$

The power delivered *to the external resistor* $\mathscr{P}_R$ is just I^2R:

$$\mathscr{P}_R = I^2R = \left(\frac{\Delta V}{R + R_i}\right) R = (\Delta V)^2 \frac{R}{(R + R_i)^2}$$

Similar to the last problem, we can maximize the power delivered to the resistor R by differentiating the power with respect to R and setting the result equal to zero:

$$\begin{aligned}
\frac{d\mathscr{P}_R}{dR} &= \frac{d}{dR}\left[(\Delta V)^2 \frac{R}{(R + R_i)^2}\right] = (\Delta V)^2 \left[\frac{1}{(R + R_i)^2} + \frac{-2R}{(R + R_i)^3}\right] = 0 \\
\Longrightarrow \qquad \frac{1}{(R + R_i)^2} &= \frac{2R}{(R + R_i)^3} \\
1 &= \frac{2R}{R + R_i} \\
R + R_i &= 2R \\
\Longrightarrow \qquad R_i &= R
\end{aligned}$$

The power is indeed extremal when the external resistor matches the internal resistance of the battery.

Again, we apply the second derivative test to see whether this is a maximum or a minimum. First, let's find the second derivative, and simplify it as much as possible.

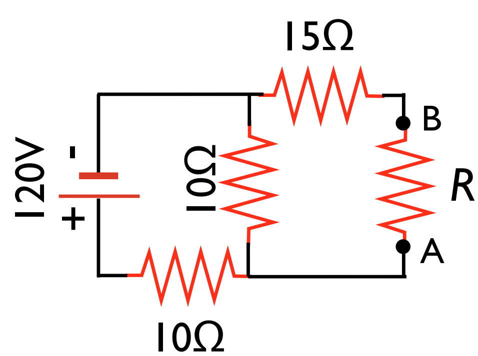
$$\begin{aligned}
\frac{d^2\mathscr{P}_R}{dR^2} &= \frac{d}{dR}\left[(\Delta V)^2\left(\frac{1}{(R+R_i)^2}-\frac{2R}{(R+R_i)^3}\right)\right]\\
&= (\Delta V)^2\left[\frac{-2}{(R+R_i)^3}-\frac{2}{(R+R_i)^3}+\frac{6R}{(R+R_i)^4}\right]\\
&= (\Delta V)^2\left[\frac{-4}{(R+R_i)^3}+\frac{6R}{(R+R_i)^4}\right]\\
&= \frac{(\Delta V)^2}{(R+R_i)^3}\left[\frac{6R}{R+R_i}-4\right]
\end{aligned}$$

We are concerned with the value of the second derivative at the point $R=R_i$, the extreme point:

$$\left.\frac{d^2\mathscr{P}_R}{dR^2}\right|_{R=R_i} = \frac{(\Delta V)^2}{(R_i+R_i)^3}\left[\frac{6R_i}{R_i+R_i}-4\right] = \frac{(\Delta V)^2}{8R_i^3}[3-4] = -\frac{(\Delta V)^2}{8R_i^3} < 0$$

The second derivative is always negative, so we have found a maximum. Thus, the power delivered to an external resistor is maximum when $R=R_i$.

3. A resistor R is to be connected across the terminals A, B of the circuit below. **(a)** For what value of R will the power dissipated in the resistor be the greatest? To answer this, construct the Thévenin equivalent circuit and then invoke the result of the previous problem. **(b)** How much power will be dissipated in R?



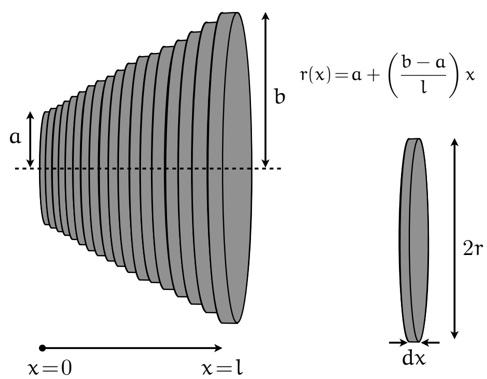
4. Two graphite rods are of equal length. One is a cylinder of radius a. The other is conical, tapering linearly from a radius a at one end to radius b at the other. Show that the end-to-end electrical resistance of the conical rod is a/b times that of the cylindrical rod. *Hint: consider the rod to be made up of thin, disk-like slices, all in series.*

The cylindrical conductor is trivial: if it is of radius a and length l, and has resistivity ρ, then

$$R_{cyl} = \frac{\rho l}{\pi a^2} \tag{1}$$

Of course, we don't know the length l or resistivity ρ, but they will not matter in the end.

What about the cone? Break the cone up into many disks of thickness dx. Stacking these disks up with increasing radius can build us a cone:



If we start out with a radius a at one end of the cone, and the other end has a radius b, then the radius as a function of position along the cone is easily determined. Let our origin $(x=0)$ be the end of the cone with radius a, and assume the cone has a total length l, same as the cylinder. Again, we will not need this length in the end, but it is convenient now. The radius at any position along the cone is then

$$r(x) = a + \left(\frac{b-a}{l}\right)x \tag{2}$$

If the current is in the x direction, then each infinitesimally thick disk is basically just a tiny segment of wire with thickness dx and cross-sectional area $\pi\left[r(x)\right]^2$. If we assume the same resistivity ρ, the resistance of each disk must be

$$dR_{cone} = \frac{\rho dx}{\pi\left[r(x)\right]^2} = \frac{\rho dx}{\pi\left(a + \left(\frac{b-a}{l}\right)x\right)^2} \tag{3}$$

The total resistance of the cone is found by integrating over all such disks, from $x=0$ to the end of the cone at $x=l$. For convenience, let $c=(b-a)/l$.

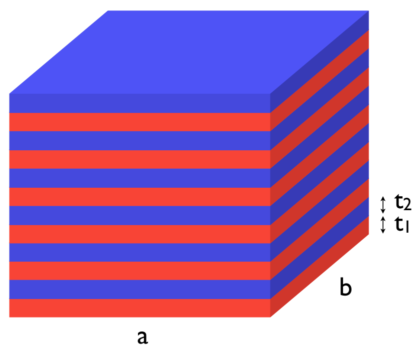
$$R_{cone} = \int dR_{cyl} = \int_0^l \frac{\rho dx}{\pi\left(a+cx\right)^2} = \frac{\rho}{\pi}\left[\frac{-1}{c\left(a+cx\right)}\right]_0^l = \frac{\rho}{\pi}\left[\frac{-1}{cb} - \frac{-1}{ca}\right] = \frac{\rho}{\pi c}\left[\frac{b-a}{ab}\right] = \frac{\rho l}{\pi ab} \tag{4}$$

Here we have a nice result: the resistance of a cone is the same as a resistance of a cylinder whose radius is the geometric mean cone's radii. That is, if we substitute $a^2 \rightarrow ab$ in our usual formula for the resistance of a cylinder, we have the result for a cone. Anyway: the desired result now follows readily,

$$\frac{R_{cone}}{R_{cyl}} = \frac{a}{b} \tag{5}$$

5. A laminated conductor was made by depositing, alternately, layers of silver 10 nm thick and layers of tin 20 nm thick. The composite material, considered on a larger scale, may be considered a homogeneous but anisotropic material with electrical conductivity $\sigma_{\perp}$ for currents perpendicular to the planes of the layers, and a different conductivity $\sigma_{||}$ for currents parallel to that plane. Given that the conductivity of silver is 7.2 times that of tin, find the ratio $\sigma_{\perp}/\sigma_{||}$.

First, let us sketch out the situation given:



see UCSD07 solns, quicker way with numbers

Now, let's solve the situation in a more general way.

We are not told how many layers of each type we have, and it will not matter in the end. For now, however, assume we have n_1 layers of tin of conductivity σ_1 and n_2 layers of silver of conductivity σ_2. Instead of conductivity, we can equivalently use resistivity ρ when it is more convenient, with $\rho = 1/\sigma$. We will also say the tin layers have thickness t_1, and the silver layers thickness t_2. The total thickness of our entire multi-layer stack is then $t_{tot} = n_1 t_1 + n_2 t_2$.

First, consider the perpendicular conductivity, the case where we pass current upward through the stack, perpendicular to the planes of the layers. When a current is flowing, electrons pass through each layer in sequence, and we can consider the stack of layers to be resistors in series. If the layers have an area of $A = ab$ (see the Figure above) and a thickness t_1 or t_2, we can readily calculate the resistance presented by a single tin or silver layer with current perpendicular to the layers:

$$R_{1,\perp} = \frac{\rho_1 t_1}{A} = \frac{t_1}{\sigma_1 A}$$
$$R_{2,\perp} = \frac{\rho_2 t_2}{A} = \frac{t_2}{\sigma_2 A}$$

For reasons that should become apparent below, it will be convenient in this case to work with the resistivity rather than the conductivity, and invert the result later. The total resistance of the stack is then just a series combination of n_1 resistors of value R_1 and n_2 resistors of value R_2:

$$R_{\text{tot},\perp} = n_1 R_{1,\perp} + n_2 R_{2,\perp} = \frac{1}{A}\left(\rho_1 t_1 n_1 + \rho_2 t_2 n_2\right)$$

If we measure the whole stack and find this resistance, we can define an effective resistivity or conductivity for the whole stack in terms of the total resistance and total thickness of the multilayer. If the resistivity of the whole stack for perpendicular currents is $\rho_\perp = 1/\sigma_\perp$, then:

$$R_{\text{tot},\perp} = \frac{\rho_\perp t_{\text{tot}}}{A} \qquad \Longrightarrow \qquad \rho_\perp = \frac{A R_{\text{tot},\perp}}{t_{\text{tot}}}$$

Now we just need to plug in what we know and simplify ...

$$\rho_\perp = \frac{A R_{\text{tot},\perp}}{t_{\text{tot}}} = \frac{A}{t_{\text{tot}}}\left[\frac{1}{A}\left(\rho_1 t_1 n_1 + \rho_2 t_2 n_2\right)\right]$$
$$\rho_\perp = \frac{\rho_1 t_1 n_1 + \rho_2 t_2 n_2}{t_{\text{tot}}} = \frac{\rho_1 t_1 n_1 + \rho_2 t_2 n_2}{n_1 t_1 + n_2 t_2}$$

We can simplify this somewhat if we realize that we have the *same* number of silver and tin layers - we are told that the layers are deposited alternatingly. If we let $n_1 = n_2 \equiv n_{bi}$, meaning we count the number of bilayers instead, then $t_{\text{tot}} = n_{bi}(t_1 + t_2)$, and

$$\rho_\perp = \frac{n_{bi}\rho_1 t_1 + n_{bi}\rho_2 t_2}{n_{bi} t_1 + n_{bi} t_2} = \frac{\rho_1 t_1 + \rho_2 t_2}{t_1 + t_2}$$

This is a nice, simple result: for current perpendicular to the planes, the effective *resistivity* is just a thickness-weighted average of the resistivities of the individual layers. Given the resistivity in the perpendicular case, we can now find the conductivity $\sigma_\perp$

$$\sigma_\perp = \frac{1}{\rho_\perp} = \frac{t_1 + t_2}{\rho_1 t_1 + \rho_2 t_2} = \frac{t_1 + t_2}{\frac{t_1}{\sigma_1} + \frac{t_2}{\sigma_2}} = \frac{\sigma_1 \sigma_2 (t_1 + t_2)}{\sigma_2 t_1 + \sigma_1 t_2}$$

As a consistency check, we can take a couple of limiting cases. First, let $\sigma_1 = \sigma_2 \equiv \sigma$. This corresponds to a homogeneous lump of a single material, and we find $\sigma_\perp = \sigma$, as expected. Next, we can check for

$\sigma_1 = 0$. In this case, one layer is not conducting at all, and since the layers are in series, this means no current flows through the stack at all, and $\sigma_\perp = 0$ as expected. Finally, we notice that the number of bilayers is irrelevant. Since the layers do not affect each other in our simple model of conduction, there is no reason to expect otherwise. So far so good. What other limiting cases can you check?

Next, let us consider current flowing parallel to the plane of the layers, from (for example) left to right in the figure above. Now the stack looks like many parallel resistors. A single tin layer of thickness t_1 and in-plane dimensions a and b now presents a resistance

$$R_{1,||} = \frac{\rho_1 a}{t_1 b} = \frac{a}{t_1 b \sigma_1}$$

Similarly, each silver layer presents a resistance

$$R_{2,||} = \frac{\rho_2 a}{t_2 b} = \frac{a}{t_2 b \sigma_2}$$

One bilayer of silver and tin means a *parallel* combination of these two resistances:

$$\frac{1}{R_{bi,||}} = \frac{1}{R_{1,||}} + \frac{1}{R_{2,||}} = \frac{b}{a}\left(t_1\sigma_1 + t_2\sigma_2\right)$$

If we have n_{bi} bilayers, then the total equivalent resistance is easily found:

$$\frac{1}{R_{tot,||}} = n_{bi}\frac{1}{R_{bi,||}} = n_{bi}\frac{b}{a}\left(t_1\sigma_1 + t_2\sigma_2\right)$$

Given the total resistance, we can now calculate the conductivity directly (in this case, first finding the resistivity does not save us any algebra), noting that the length of the whole stack along the direction of the current is just a, and the cross-sectional area is $bt_{tot} = bn_{bi}\left(t_1 + t_2\right)$:

$$\sigma_{||} = \frac{1}{\rho_{||}} = \frac{a}{n_{bi}\left(t_1 + t_2\right) bR_{tot,||}} = \frac{a}{n_{bi}\left(t_1 + t_2\right) b}\left(\frac{b}{a}\right) n_{bi}\left(t_1\sigma_1 + t_2\sigma_2\right) = \frac{t_1\sigma_1 + t_2\sigma_2}{t_1 + t_2}$$

Again, a sensible result: the effective *conductivity* for current parallel to the planes is just a thickness-weighted average of the conductivities of the individual layers. Again, you can convince yourself with a couple of limiting cases that this result makes some sense.

Now that we have both parallel and perpendicular conductivities, we can easily find the anisotropy $\sigma_\perp/\sigma_{||}$.

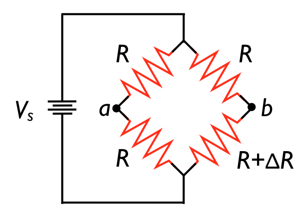
$$\sigma_\perp/\sigma_{||} = \frac{\rho_{||}}{\rho_\perp} = \frac{\sigma_1\sigma_2\left(t_1+t_2\right)}{\sigma_2 t_1+\sigma_1 t_2}\frac{t_1+t_2}{\sigma_1 t_1+\sigma_2 t_2} = \frac{\sigma_1\sigma_2\left(t_1+t_2\right)^2}{\left(t_1\sigma_1+t_2\sigma_2\right)\left(t_1\sigma_2+t_2\sigma_1\right)}$$

Finally, we are given that the conductivity of silver is 7.2 times that of tin, and the tin layers' thickness is twice that of the silver. Thus, $t_1 = 2t_2$ and $\sigma_2 = 7.2\sigma_1$. The actual values and units do not matter, as this is a dimensionless ratio (you should verify this fact ...), and you should find $\sigma_\perp/\sigma_{||} \approx 0.457$.

And, once again, you can check that for $\sigma_1 = \sigma_2$, we have $\sigma_\perp/\sigma_{||} = 1$, as it must if both materials are the same.

□ **6.** The circuit at right is known as a *Wheatstone Bridge*, and it is a useful circuit for measuring small changes in resistance. Perhaps you can figure out why. Three of the four branches on our bridge have identical resistance R, but the fourth has a slightly different resistance, ΔR more than the other branches, such that its total resistance is $R + \Delta R$.

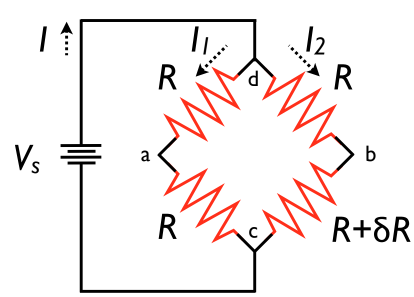
In terms of the source voltage V_s, base resistance R and change in resistance ΔR, what is the potential difference between points a and b? You may assume the voltage source and wires are perfect (no internal resistance and no voltage drop, respectively).



Wheatstone Bridge

Label the nodes on the bridge a-d, as shown in the figure below, and let a current I_1 flow from point d through a to c, and a current I_2 flow from d through b to c.

Looking more carefully at the bridge, we notice that it is nothing more than two sets of series resistors, connected in parallel with each other. This immediately means that the voltage drop across the left side of the bridge, following nodes d→a→c, must be the same as the voltage drop across the right side of the bridge, following nodes d→b→c. Both are ΔV_{dc}, and both must be the same as the source voltage:



Labeling notes and currents in the Wheatstone Bridge

$\Delta V_{dc} = V_s$. If we can find the current in each resistor, then with the known source potential difference we will know the voltage at any point in the circuit we like, and finding ΔV_{ab} is no problem.

Let the current from the source V_s be I. This current I leaving the source will at node a split into the separate currents I_1 and I_2; conservation of charge requires $I = I_1 + I_2$. At node c, the currents recombine into I. On the leftmost branch of the bridge, the current I_1 creates a voltage drop I_1R across each resistor. Similarly, on the rightmost branch of the bridge, the resistor R has a voltage drop I_2R and the lower resistor has a voltage drop $I_2(R + \delta R)$. Equating the total voltage drop on each branch of the bridge:

$$
\begin{aligned}
& V_s = I_1R + I_1R = I_2R + I_2(R + \delta R) \\
\implies \quad & I_1 = \frac{V_s}{2R} \\
& I_2 = \frac{V_s}{2R + \delta R}
\end{aligned}
$$

Now that we know the currents in terms of known quantities, we can find ΔV_{ab} by "walking" from point a to point b and summing the changes in potential difference. Starting at node a, we move toward node d *against* the current I_1, which means we *gain* a potential difference I_1R. Moving from node d to node b, we move *with* the current I_2, which means we *lose* a potential difference I_2R. Thus, the total potential difference between points a and b must be

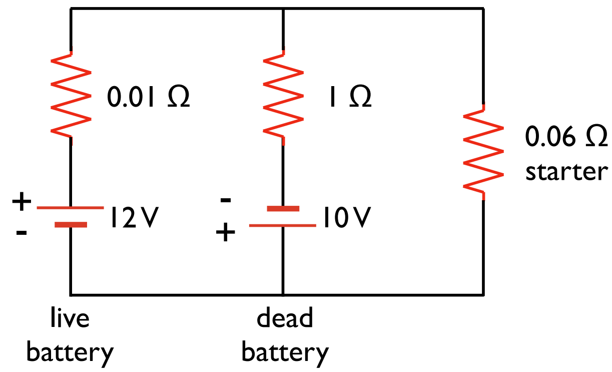
$$
\begin{aligned}
\Delta V_{ab} &= I_1R - I_2R = R(I_1 + I_2) = R\left(\frac{V_s}{2R} - \frac{V_s}{2R + \delta R}\right) \\
\Delta V_{ab} &= V_s\left(\frac{1}{2} - \frac{R}{R + \delta R}\right) = V_s\left(\frac{\delta R}{4R + 2\delta R}\right)
\end{aligned}
$$

If the change in resistance δR is small compared to R ($\delta R \ll R$), the term in the denominator can be approximated $4R + \delta R \approx 4R$, and we have

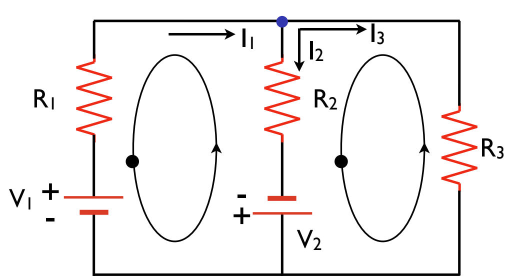
$$\Delta V_{ab} = \frac{1}{4}V_s\left(\frac{\delta R}{R}\right) \qquad (\delta R \ll R)$$

Thus, for small changes in resistance, the voltage measured across the bridge is directly proportional to the change in resistance, which is the basic utility of this circuit: it allows one to measure small changes on top of a large 'base' resistance. Fundamentally, it is a *difference* measurement, meaning that one directly measures *changes* in the quantity of interest, rather than measuring the whole thing and trying to uncover subtle changes. This behavior is very useful for, e.g., strain gauges, temperature sensors, and many other devices.

7. A dead battery is charged by connecting it to the live battery of another car with jumper cables (see below). Determine the current in the starter and in the dead battery.



Since this circuit has several branches and multiple batteries, we cannot reduce it by using our rules of series and parallel resistors - we have to use our general circuit rules (Kirchhoff's rules). In order to do that, we first need to assign currents in each branch of the circuit. It doesn't matter what directions we choose at all, assigning directions is just to define what is, relatively speaking, positive and negative. If we choose the direction for one current incorrectly, we will get a negative number for that current to let us know. Below, we choose initial currents I_1, I_2, and I_3 in each branch of the circuit.



Here we have also labeled each component symbolically to make the algebra a bit easier to sort out – we don't want to just plug in numbers right away, or we'll make a mess of things. Note that since we have three unknowns – the three currents – so we will need three equations to solve this problem completely. We have three possible loops (left, right, outer), which gives us two equations, and two junctions, which gives us one more. If we have N loops or N junctions, we get $N - 1$ equations from either.

Now we are ready to apply the rules. First, conservation of charge (the "junction rule"). We have only two junctions in this circuit, in the center at the top and bottom where three wires meet. The junction rule basically states that the current into a junction (or node) must equal the current out. In the case of the upper node, this means:

$$I_1 = I_2 + I_3 \tag{6}$$

You can easily verify that the lower node gives you the same equation. Next, we can apply conservation of energy (the "loop rule"). There are three possible loops we can take: the rightmost one containing R_3 and R_2, the leftmost one containing R_1 and R_2, and the outer perimeter (containing R_1 and R_3). We only need to work through two of them - we have already one equation above, and we only need two more. Somewhat arbitrarily, we will pick the right and left side loops.

First, the left-hand side loop. Start just above the live battery V_1, and walk *clockwise* around the loop. We cross the battery from negative to positive for a *gain* in potential energy, and we cross R_1 and R_3 in the direction of current flow for a *loss* of potential energy. These three have to sum to zero for a closed loop:

$$-I_1R_1 - I_2R_2 + V_2 + V_1 = 0 \tag{7}$$

Next, the right-hand side loop. Again, start just above the battery (V_2 this time), and walk *clockwise* around the loop. Now we cross R_2 against the current and R_3 with the current for a gain and loss of voltage, respectively, and then cross the battery in the wrong direction for a potential drop:

$$I_2R_2 - I_3R_3 - V_2 = 0 \tag{8}$$

Now we have three equations and three unknowns, and we are left with the pesky problem of solving them for the three currents. There are many ways to do this, we will illustrate two of them. Before we get started, let us repeat the three questions in a more consistent form.

$$\begin{aligned}
I_1 - I_2 - I_3 &= 0 \\
R_1I_1 + R_2I_2 &= V_1 + V_2 \\
R_2I_2 - R_3I_3 &= V_2
\end{aligned}$$

The first way we can proceed is by substituting the first equation into the second:

$$V_1 + V_2 = R_1I_1 + R_2I_2 = R_1\left(I_2 + I_3\right) + R_2I_2 = \left(R_1 + R_2\right)I_2 + R_1I_3$$

Now our three equations look like this:

$$\begin{aligned} I_1 - I_2 - I_3 &= 0 \\ (R_1 + R_2)\, I_2 + R_1 I_3 &= V_1 + V_2 \\ R_2 I_2 - R_3 I_3 &= V_2 \end{aligned}$$

The last two equations now contain only I_1 and I_2, so we can solve the third equation for I_2:

$$I_2 = \frac{I_3 R_3 + V_2}{R_2}$$

...and plug it in to the second one:

$$\begin{aligned} V_1 + V_2 = (R_1 + R_2)\, I_2 + R_1 I_3 &= (R_1 + R_2)\left(\frac{I_3 R_3 + V_2}{R_2}\right) + R_1 I_3 \\ R_2 (V_1 + V_2) &= R_3 (R_1 + R_2)\, I_3 + V_2 (R_1 + R_2) + R_1 R_2 I_3 \\ V_1 R_2 - V_2 R_1 &= (R_1 R_3 + R_2 R_3 + R_1 R_2)\, I_3 \\ \Longrightarrow \qquad I_3 &= \frac{R_2 V_1 - R_1 V_2}{R_1 R_2 + R_2 R_3 + R_1 R_3} \approx 169\,\text{A} \end{aligned}$$

There is a sort of pleasing symmetry to the analytical answer. Now that you know I_3, you can plug it in the expression for I_2 above, you should find $I_2 \approx 20\,\text{A}$, and similarly you can find $I_1 \approx 189\,\text{A}$

Optional: There is one more way to solve this set of equations using matrices and Cramer's rule,[ii] if you are familiar with this technique. If you are not familiar with matrices, you can skip to the next problem - you are not required or necessarily expected to know how to do this. First, write the three equations in matrix form:

$$\begin{bmatrix} R_1 & R_2 & 0 \\ 0 & R_2 & -R_3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 + V_2 \\ V_2 \\ 0 \end{bmatrix}$$

$$aI = V$$

The matrix a times the column vector I gives the column vector V, with the matrices defined thusly:

$$a = \begin{bmatrix} R_1 & R_2 & 0 \\ 0 & R_2 & -R_3 \\ 1 & -1 & -1 \end{bmatrix} \qquad I = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \qquad V = \begin{bmatrix} V_1 + V_2 \\ V_2 \\ 0 \end{bmatrix}$$

[ii] See 'Cramer's rule' in the Wikipedia to see how this works.

Now we can use the determinant of the matrix a with Cramer's rule to find the currents. For each current, we construct a new matrix, which is the same as the matrix a except that the column of a corresponding to that current is replaced the column vector V. Thus, for I_1, we replace column 1 in a with V, and for I_2, we replace column 2 in a with V. We find the current then by calculating the determinant of the new matrix and dividing it by $\det a$. Below, we have highlighted the columns in a which have been replaced to make this more clear:

$$I_1 = \frac{\begin{vmatrix} V_1+V_2 & R_2 & 0 \\ V_2 & R_2 & -R_3 \\ 0 & -1 & -1 \end{vmatrix}}{\det a} \qquad I_2 = \frac{\begin{vmatrix} R_1 & V_1+V_2 & 0 \\ 0 & V_2 & -R_3 \\ 1 & 0 & -1 \end{vmatrix}}{\det a} \qquad I_3 = \frac{\begin{vmatrix} R_1 & R_2 & V_1+V_2 \\ 0 & R_2 & V_2 \\ 1 & -1 & 0 \end{vmatrix}}{\det a}$$

Now we need to calculate the determinant of each new matrix, and divide that by $\det a$.[iii] First, the determinant of a.

$$\det a = -R_1R_2 - R_1R_3 - R_2R_3 = -\left(R_1R_2 + R_2R_3 + R_1R_3\right)$$

We can now find the currents readily from the determinants of the modified matrices and $\det a$ we just found. Thankfully, the matrices have enough zeros that it is relatively easy. In case your memory is rusty, here is the determinant of an arbitrary 3×3 matrix:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \left(a_1b_2c_3 - a_1b_3c_2\right) + \left(a_2b_3c_1 - a_2b_1c_3\right) + \left(a_3b_1c_2 - a_3b_2c_1\right)$$

Given this,

$$I_1 = \frac{-\left(V_1+V_2\right)R_2 - R_3\left(V_1+V_2\right) + R_2V_2}{-\left(R_1R_2 + R_2R_3 + R_1R_3\right)} = \frac{\left(V_1+V_2\right)R_3 + R_2V_1}{R_1R_2 + R_2R_3 + R_1R_3} \approx 189\,\text{A}$$

$$I_2 = \frac{-R_1V_2 - \left(V_1+V_2\right)R_3}{-\left(R_1R_2 + R_2R_3 + R_1R_3\right)} = \frac{\left(V_1+V_2\right)R_3 + R_1V_2}{R_1R_2 + R_2R_3 + R_1R_3} \approx 20\,\text{A}$$

$$I_3 = \frac{R_1V_2 + R_2V_2 - \left(V_1+V_2\right)R_2}{-\left(R_1R_2 + R_2R_3 + R_1R_3\right)} = \frac{R_2V_1 - R_1V_2}{R_1R_2 + R_2R_3 + R_1R_3} \approx 169\,\text{A}$$

These are the same results you would get by continuing on with the previous 'plug-n-chug' method. Both numerically and symbolically, we can see from the above that $I_1 = I_2 + I_3$:

[iii] Again, the Wikipedia entry for 'determinant' is quite instructive.

$$I_2 + I_3 = \frac{(V_1 + V_2)\,R_3 + R_1V_2 + R_2V_1 - R_1V_2}{R_1R_2 + R_2R_3 + R_1R_3} = \frac{(V_1 + V_2)\,R_3 + R_2V_1}{R_1R_2 + R_2R_3 + R_1R_3} = I_1$$

8. A hair dryer intended for travelers operates at 115 V and also at 230 V. A switch on the dryer adjusts the dryer for the voltage in use. At each voltage, the dryer delivers 1000 W of heat.

(a) What must the resistance of the heating coils be for each voltage?
(b) For such a dryer, sketch a circuit consisting of two identical heating coils connected to a switch and the power outlet. Opening and closing the switch should give the proper resistance for each voltage.
(c) What is the current in the heating elements at each voltage?

9. Two capacitors, one charged and the other uncharged, are connected in parallel. **(a)** Prove that when equilibrium is reached, each carries a fraction of the initial charge equal to the ratio of its capacitance to the sum of the two capacitances. **(b)** Show that the final energy is less than the initial energy, and derive a formula for the difference in terms of the initial charge and the two capacitances.

This problem is easiest to start if you approach it from a conservation of energy & charge point of view. We have two capacitors. Initially, one capacitor stores a charge Q_{1i}, while the other is empty, $Q_{2i} = 0$. After connecting them together in parallel, some charge leaves the first capacitor and goes to the second, leaving the two with charges Q_{1f} and Q_{2f}, respectively. Now, since there were no sources hooked up, and we just have the two capacitors, the total amount of charge must be the same before and after we hook them together:

$$\begin{aligned} Q_i &= Q_f \\ Q_{1i} + Q_{2i} &= Q_{1f} + Q_{2f} \qquad (\text{given } Q_{2i} = 0) \\ Q_{1i} &= Q_{1f} + Q_{2f} \end{aligned}$$

We also know that if two capacitors are connected in parallel, they will have the same voltage ΔV across them:

$$\Delta V_f = \frac{Q_{1f}}{C_1} = \frac{Q_{2f}}{C_2}$$

The fraction of the total charge left on the first capacitor can be found readily combining what we have:

$$\frac{Q_{1f}}{Q_i} = \frac{Q_{1f}}{Q_{1i}} = \frac{Q_{1f}}{Q_{1f} + Q_{2f}} = \frac{Q_{1f}}{Q_{1f} + \frac{C_2}{C_1}Q_{1f}} = \frac{C_1Q_{1f}}{C_1Q_{1f} + C_2Q_{1f}} = \frac{C_1}{C_1 + C_2}$$

The second capacitor must have the rest of the charge:

$$\frac{Q_{2f}}{Q_i} = 1 - \frac{C_1}{C_1 + C_2} = \frac{C_2}{C_1 + C_2}$$

That was charge conservation. We can also apply energy conservation, noting that the energy of a charged capacitor is $Q^2/2C$:

$$\begin{aligned} E_i &= E_f \\ \frac{Q_{1i}^2}{2C_1} &= \frac{Q_{1f}^2}{2C_1} + \frac{Q_{2f}^2}{2C_2} \end{aligned}$$

The final energy can be simplified using the result of the first part of the problem - we note that $Q_{1f} = Q_i C_1/(C_1 + C_2)$ and $Q_{2f} = Q_i C_2/(C_1 + C_2)$

$$\begin{aligned} E_f &= \frac{Q_{1f}^2}{2C_1} + \frac{Q_{2f}^2}{2C_2} \\ &= \left(\frac{Q_i C_1}{C_1 + C_2}\right)^2 \frac{1}{2C_1} + \left(\frac{Q_i C_2}{C_1 + C_2}\right)^2 \frac{1}{2C_2} \\ &= \frac{Q_i^2 C_1}{2(C_1 + C_2)^2} + \frac{Q_i^2 C_2}{2(C_1 + C_2)^2} \\ &= \frac{Q_i^2 (C_1 + C_2)}{2(C_1 + C_2)^2} = \frac{Q_i^2}{2(C_1 + C_2)} \\ &= \frac{Q_i^2}{2C_1}\left(\frac{C_1}{C_1 + C_2}\right) = E_i \left(\frac{C_1}{C_1 + C_2}\right) \end{aligned}$$

Thus, the final energy will be less than the initial energy, by a factor $C_1/(C_1 + C_2) < 1$.