UNIVERSITY OF ALABAMA
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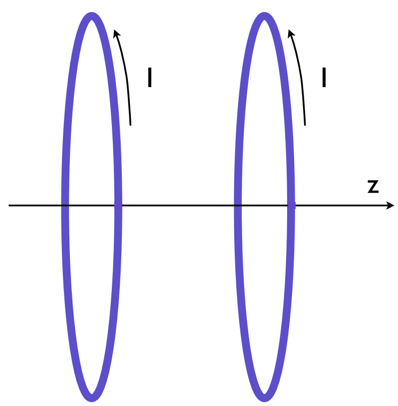
Problem Set 5: Solutions

1. A Helmholtz coil consists of two identical circular coils of radius R separated by a distance equal to their radii, as shown at below. Each carries current I in the same direction. **(a)** Find the field at any point along the coil axis between the two coils. **(b)** Show that $\partial B/\partial z$ vanishes at the midpoint. **(c)** Show that $\partial^2 B/\partial z^2$ also vanishes at the midpoint of the two coils.



We first want to find the field at an *arbitrary* point z somewhere between the two coils, then find dB/dz and evaluate it at the midpoint. We don't just want the field at the midpoint; we need its spatial variation to find the derivative(s) and only *then* do we evaluate our quantities at the midpoint between the two coils.

Let $z=0$ at the intersection of the plane of the bottom coil and the z axis. The field from the bottom coil at an arbitrary point a distance z along the axis due to the bottom coil is that of a single current loop centered on the origin, which we have already established:

$$B = \frac{\mu_o I}{2}\frac{R^2}{(z^2+R^2)^{3/2}} \quad \text{(single loop)}$$

At a position z, since the separation of the coils is R, we are a distance $R-z$ from the upper coil. We need only replace z with $R-z$ in the expression above to find the field from the upper coil at a distance $z<R$ from the bottom coil. Since the currents are in the same directions for both coils, the magnetic fields are in the same direction, and we may just add them together:

$$B_{tot} = B_{lower} + B_{upper} = \frac{\mu_o I}{2}\frac{R^2}{(z^2+R^2)^{3/2}} + \frac{\mu_o I}{2}\frac{R^2}{\left[(R-z)^2+R^2\right]^{3/2}}$$

Now we only need calculate dB/dz:

$$\frac{\partial B_{tot}}{\partial t} = \frac{\mu_o I R^2}{2}\left[\frac{d}{dz}\frac{1}{\left(z^2+R^2\right)^{3/2}} + \frac{d}{dz}\frac{1}{\left(\left(R-z\right)^2+R^2\right)^{3/2}}\right]$$

$$= \frac{\mu_o I R^2}{2}\left[\frac{-\frac{3}{2}\left(2z\right)}{\left(z^2+R^2\right)^{5/2}} + \frac{-\frac{3}{2}\left(2z-2R\right)}{\left(\left(R-z\right)^2+R^2\right)^{5/2}}\right]$$

$$= \frac{\mu_o I R^2}{2}\left[\frac{-3z}{\left(z^2+R^2\right)^{5/2}} + \frac{3R-3z}{\left(\left(R-z\right)^2+R^2\right)^{5/2}}\right]$$

Evaluating this at point P, where $z = R/2$,

$$\left.\frac{\partial B_{tot}}{\partial t}\right|_{z=\frac{R}{2}} = \frac{\mu_o I R^2}{2}\left[\frac{-3\frac{R}{2}}{\left(\frac{R^2}{4}+R^2\right)^{5/2}} + \frac{3R-3\frac{R}{2}}{\left(\left(R-\frac{R}{2}\right)^2+R^2\right)^{5/2}}\right]$$

$$= \frac{\mu_o I R^2}{2}\left[\frac{-3R}{2\left(\frac{5R^2}{4}\right)^{5/2}} + \frac{3R}{2\left(\frac{5R^2}{4}\right)^{5/2}}\right] = 0$$

Indeed, the derivative of the field with respect to position is zero, implying high uniformity near the center of the Helmholtz arrangement. Repeat to find the second spatial derivative:

$$\frac{\partial^2 B_{tot}}{\partial z^2} = \frac{\mu_o I R^2}{2}\left[\frac{-3}{\left(z^2+R^2\right)^{5/2}} + \frac{-3z\left(-\frac{5}{2}\right)\left(2z\right)}{\left(z^2+R^2\right)^{7/2}} + \frac{3}{\left(\left(R-z\right)^2+R^2\right)^{5/2}} + \frac{3\left(z-R\right)\left(-\frac{5}{2}\right)\left(2z-2R\right)}{\left(\left(R-z\right)^2+R^2\right)^{7/2}}\right]$$

$$= \frac{\mu_o I R^2}{2}\left[\frac{15z^2}{\left(z^2+R^2\right)^{7/2}} - \frac{3}{\left(z^2+R^2\right)^{5/2}} - \frac{15\left(z-R\right)^2}{\left(\left(R-z\right)^2+R^2\right)^{7/2}} + \frac{3}{\left(\left(R-z\right)^2+R^2\right)^{5/2}}\right]$$

Evaluating at $z = R/2$,

$$\left.\frac{\partial^2 B_{tot}}{\partial z^2}\right|_{z=\frac{R}{2}} = \frac{\mu_o I R^2}{2}\left[\frac{15\left(\frac{R^2}{4}\right)}{\left(\left(\frac{R^2}{4}\right)+R^2\right)^{7/2}} - \frac{3}{\left(\left(\frac{R^2}{4}\right)+R^2\right)^{5/2}} - \frac{15\left(\frac{R^2}{4}\right)}{\left(\left(\frac{R^2}{4}\right)+R^2\right)^{7/2}} + \frac{3}{\left(\left(\frac{R^2}{4}\right)+R^2\right)^{5/2}}\right] = 0$$

One can also show that d^3B/dz^3 is zero as well (but it is messy), meaning the field is extremely uniform over a reasonably large volume between the two coils.

Incidentally, the field midway between the two coils, with $z = R/2$ is $B = 8\mu_o I/5^{3/2}R$, about 1.4 times the field one would get directly at the center of a single coil. Not only is the field more uniform, it is larger. How convenient!

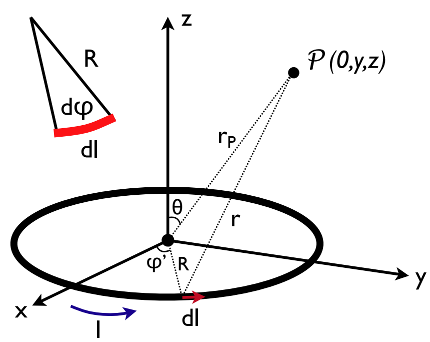
2. A current loop of radius R lies in the xy plane carrying a steady current I. Show that for distances large compared to R the resulting magnetic field may be written

$$B_r = \frac{\mu_0 |\vec{\mu}| \cos\theta}{2\pi r^3} \tag{1}$$

$$B_\theta = \frac{\mu_0 |\vec{\mu}| \sin\theta}{4\pi r^3} \tag{2}$$

Here r is the distance from the center of the loop, and θ is relative to the z axis (which is perpendicular to the plane of the loop).

See the figure below. Let the loop lie in the xy plane, and let the vector $\vec{r}$ point from the source point to the field point $\mathscr{P}$. We will restrict ourselves field points in the yz plane (i.e., $x = 0$). Since the problem is rotationally symmetric about the z axis, this is sufficient to find the field everywhere.
At the source point, a segment of the current loop $d\vec{l}$ is located at $(x, y, z) = (R\cos\varphi, R\sin\varphi, 0)$ and φ runs from 0 to 2π. Let the vector $\vec{r}_p$ point from the origin, at the center of the current loop, to the field point $\mathscr{P}$ lying at $(0, y, z)$.



We can, in principle, find the magnetic field at an arbitrary point with the Biot-Savart law using the distance $\vec{r}$ from a current element $d\vec{l}$ to the source point $\mathscr{P}$

$$\vec{B} = \frac{\mu_o I}{4\pi} \int \frac{d\vec{l'} \times \vec{r}}{|\vec{r}|^3} \tag{3}$$

However, we have a bit of geometry to overcome before this is of any use. First, it is useful define an vector $\vec{R}$ which lies in the xy plane and points from the origin to the element $d\vec{l}$. This is nothing more than the radial vector in cylindrical coordinates, and it is easy to construct given the location of $d\vec{l}$:

$$\vec{R} = R\cos\varphi\,\hat{\mathbf{x}} + R\sin\varphi\,\hat{\mathbf{y}} \tag{4}$$

Actually, from a certain point of view we'll be mixing cylindrical and spherical coordinates freely, which might be disconcerting. From another point of view it is irrelevant so long as we are careful to put everything back into cartesian coordinates before attempting any vector products. Anyway: the element $d\vec{l}$ is just the arclength defined by R and $d\varphi$, which is also easily defined in terms of the angular unit vector $\hat{\boldsymbol{\varphi}}$ and its cartesian equivalent.

$$d\vec{l'} = R\,d\varphi\,\hat{\boldsymbol{\varphi}} = -R\sin\varphi\,d\varphi\,\hat{\mathbf{x}} + R\cos\varphi\,d\varphi\,\hat{\mathbf{y}} \tag{5}$$

A bit more tricky in this case is to find the vector $\vec{r}$. Given a vector locating the field point $\vec{r}_p$ and $\vec{R}$, the relative position vector is

$$\vec{r} = \vec{r}_p - \vec{R} = -R\cos\varphi\,\hat{\mathbf{x}} + (y - R\sin\varphi)\;\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} \tag{6}$$

You can verify this in one of two ways. First, since we have $\vec{R}$ in component form above, one can just subtract it from $\vec{r}_p$ which is the standard radial vector in spherical coordinates. Second, from the geometry of the figure above, you can find the components along the x, y, and z axis with a bit of trigonometry. Either way, the result is the same. The magnitude of the separation vector is thus

$$|\vec{r}|^2 = R^2\cos^2\varphi + y^2 - 2yR\sin\varphi + R^2\sin^2\varphi + z^2 = R^2 + y^2 + z^2 - 2Ry\sin\varphi \tag{7}$$

Next, we can calculate the cross product $d\vec{l'} \times \vec{r}$ and simplify it a bit:

$$d\vec{l'} \times \vec{r} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -R\sin\varphi\,d\varphi & R\cos\varphi\,d\varphi & 0 \\ -R\cos\varphi & (y - R\sin\varphi) & z \end{vmatrix} \tag{8}$$

$$= Rz\cos\varphi\,d\varphi\,\hat{\mathbf{x}} + Rz\sin\varphi\,d\varphi\,\hat{\mathbf{y}} + \left(-Ry\sin\varphi\,d\varphi + R^2\,d\varphi\right)\hat{\mathbf{z}} \tag{9}$$

$$= R\,d\varphi\left[z\cos\varphi\,\hat{\mathbf{x}} + z\sin\varphi\,\hat{\mathbf{y}} + (R - y\sin\varphi)\;\hat{\mathbf{z}}\right] \tag{10}$$

That's it: all we have to do next is set up the integral. This is where the "fun" starts of course. And by "fun" we mean "pain" as usual. The integral itself is over all the $d\vec{l'}$ making up the current loop, which in this case means an integral over $d\varphi$ from 0 to 2π. Just plugging everything into the Biot-Savart law, we have

$$\vec{B} = \frac{\mu_o I}{4\pi}\int_0^{2\pi} \frac{R\,d\varphi\left[z\cos\varphi\,\hat{\mathbf{x}} + z\sin\varphi\,\hat{\mathbf{y}} + (R - y\sin\varphi)\;\hat{\mathbf{z}}\right]}{\left[R^2 + y^2 + z^2 - 2Ry\sin\varphi\right]^{3/2}} \tag{11}$$

Right off the bat, we can show that the x component vanishes, since it is a pure differential:[i]

$$B_x = \frac{\mu_o IRz}{4\pi}\int_0^{2\pi} \frac{\cos\varphi\,d\varphi}{\left(R^2 + y^2 + z^2 - 2Ry\sin\varphi\right)^{3/2}} = \frac{\mu_o Iz}{4\pi y}\frac{1}{\sqrt{R^2 + y^2 + z^2 - 2Ry\sin\varphi}}\Bigg|_0^{2\pi} = 0 \tag{12}$$

This simplifies things a bit:

$$\vec{B} = \frac{\mu_o I}{4\pi}\int_0^{2\pi} \frac{R\,d\varphi\left[z\sin\varphi\,\hat{\mathbf{y}} + (R - y\sin\varphi)\;\hat{\mathbf{z}}\right]}{\left[R^2 + y^2 + z^2 - 2Ry\sin\varphi\right]^{3/2}} \tag{13}$$

Of course, we have one more nagging point: we actually want the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ components of the field, not the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ components. No matter: first, just remember that since we have restricted the field point to the yz plane (so $x=0$)

[i] Consider a change of variable $u = R^2 + y^2 + z^2 - 2yR\sin\varphi$ if you don't see this right away.

$$B_r = B_y \sin\theta + B_z \cos\theta \tag{14}$$

$$B_\theta = B_y \cos\theta - B_z \sin\theta \tag{15}$$

$$\hat{\mathbf{r}} = \sin\theta\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \tag{16}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \tag{17}$$

$$r^2 = y^2 + z^2 \tag{18}$$

Using these relationships and Eq. 13, we can find the radial component without too much trouble. That is, if we remember that $z\sin\theta = y\cos\theta$, $r^2 = y^2 + z^2$, and $y = r\sin\theta$ ...

$$B_r = \frac{\mu_o I}{4\pi} \int_0^{2\pi} \frac{R\,d\varphi\,[z\sin\varphi\sin\theta + R\cos\theta - y\sin\varphi\cos\theta]}{[R^2 + y^2 + z^2 - 2Ry\sin\varphi]^{3/2}} = \frac{\mu_o I}{4\pi} \int_0^{2\pi} \frac{R\,d\varphi\,[\sin\varphi\,(z\sin\theta - y\cos\theta) + R\cos\theta]}{[R^2 + r^2 - 2rR\sin\theta\sin\varphi]^{3/2}}$$

$$B_r = \frac{\mu_o I R^2 \cos\theta}{4\pi} \int_0^{2\pi} \frac{d\varphi}{[R^2 + r^2 - 2rR\sin\theta\sin\varphi]^{3/2}} \tag{19}$$

The angular component is not much more work, if we this time note that $z\cos\theta + y\sin\theta = z^2/r + y^2/r = r$

$$B_\theta = \frac{\mu_o I}{4\pi} \int_0^{2\pi} \frac{R\,d\varphi\,[z\sin\varphi\cos\theta - R\sin\theta + y\sin\varphi\sin\theta]}{[R^2 + r^2 - 2rR\sin\theta\sin\varphi]^{3/2}} = \frac{\mu_o IR}{4\pi} \int_0^{2\pi} \frac{d\varphi\ \sin\varphi\,(z\cos\theta + y\sin\theta) - d\varphi\ \sin\theta}{[R^2 + r^2 - 2rR\sin\theta\sin\varphi]^{3/2}}$$

$$B_\theta = \frac{\mu_o IR}{4\pi} \int_0^{2\pi} \frac{d\varphi\ (r\sin\varphi - R\sin\theta)}{[R^2 + r^2 - 2rR\sin\theta\sin\varphi]^{3/2}} \tag{20}$$

At this point, extreme displeasure sets in. It is not that our expressions for B_r and B_θ seem unsightly: that much we could live with. What is more hideous, on closer inspection, is that both components involve elliptical integrals, which have no analytic solution. This is useless to us, so we will need an approximation. The most sensible thing is to consider large distances from the loop, such that $r \gg R$. This will simplify the nasty denominator in the integral, which is the real source of our trouble. Considering points for which $R/r \ll 1$,

$$\frac{1}{[R^2 + r^2 - 2rR\sin\theta\sin\varphi]^{3/2}} = \frac{1}{r^3} \frac{1}{\left[1 + \frac{R^2}{r^2} - \frac{2R}{r}\sin\varphi\sin\theta\right]^{3/2}} \approx \frac{1}{r^3}\left[1 - \frac{3R^2}{2r^2} + \frac{3R}{r}\sin\varphi\sin\theta\right] \tag{21}$$

This approximation gives an analytically solvable integral for B_r:

$$B_r \approx \frac{\mu_o I R^2 \cos\theta}{4\pi} \int_0^{2\pi} \frac{d\varphi}{r^3}\left[1 - \frac{3R^2}{2r^2} + \frac{3R}{r}\sin\varphi\sin\theta\right] = \frac{\mu_o I R^2 \cos\theta}{4\pi r^3}\left[\varphi\left(1 - \frac{3R^2}{2r^2}\right) - \frac{3R}{r}\cos\varphi\sin\theta\right]_0^{2\pi}$$

$$B_r \approx \frac{\mu_o I R^2 \cos\theta}{4\pi r^3}(2\pi)\left(1 - \frac{3R^2}{2r^2}\right) \tag{22}$$

In the limit that $R/r \ll 1$, we may safely neglect terms which are second order in R/r. If we recognize the magnetic moment of the loop $|\vec{\mu}|=\pi R^2 I$, we are halfway home:

$$B_r \approx \frac{\mu_o|\vec{\mu}|\cos\theta}{2\pi r^3} \qquad (r \gg R) \tag{23}$$

We can just use the same approximation for the radial component, this time with the benefit of hindsight: the second-order term in R/r can be neglected, and any terms involving only $\sin\varphi$ will integrate to zero over $[0, 2\pi]$

$$B_\theta \approx \frac{\mu_o IR}{4\pi r^3}\int_0^{2\pi} (r\sin\varphi - R\sin\theta)\left[1 - \frac{3R^2}{2r^2} + \frac{3R}{r}\sin\varphi\sin\theta\right] d\varphi$$

$$B_\theta \approx \frac{\mu_o IR}{4\pi r^3}\int_0^{2\pi} r\sin\varphi\left[1 - \frac{3R^2}{2r^2}\right] + 3R\sin^2\varphi\sin\theta - R\sin\theta\left(1 - \frac{3R^2}{r^2}\right) - \frac{3R^2}{2r}\sin\varphi\sin^2\theta\, d\varphi$$

$$B_\theta \approx \frac{\mu_o IR}{4\pi r^3}\int_0^{2\pi} r\sin\varphi\left[1 - \frac{3R^2}{2r^2} - \frac{3R^2}{2r^2}\sin^2\theta\right] + 3R\sin^2\varphi\sin\theta - R\sin\theta\left(1 - \frac{3R^2}{r^2}\right) d\varphi$$

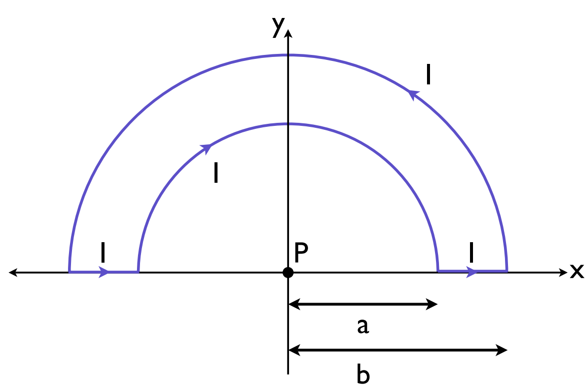
$$B_\theta \approx \frac{\mu_o IR}{4\pi r^3}\left[\frac{3}{2}R\sin\theta\left(\varphi - \sin\varphi\cos\varphi\right) - R\varphi\sin\varphi\right]_0^{2\pi} = \frac{\mu_o IR}{4\pi r^3}\left[3\pi R\sin\theta - 2\pi R\sin\theta\right]$$

$$B_\theta \approx \frac{\mu_o|\vec{\mu}|\sin\theta}{4\pi r^3} \qquad (r \gg R) \tag{24}$$

In total, we may write the field as

$$\vec{B} = \frac{\mu_o|\vec{\mu}|}{4\pi r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) \tag{25}$$

3. Find the magnetic field at point P due to the current distribution shown below. *Hint: Break the loop into segments, and use superposition.*



The easiest way to do solve this is by superposition. Since the magnetic field obeys superposition, we can consider our odd current loop above to be the same as two semicircles plus two small straight segments. We know that the magnetic field at the center of a circular loop of radius r carrying a current I is

$$B = \frac{\mu_o I}{2r} \qquad \text{(loop radius r)}$$

If you did not remember this, it is easily derived from the Biot-Savart law ... or you could notice that problem 5 actually gives you this expression if you set $z = 0$. As a quick reminder, for a circle of radius r, our infinitesimal length element dl is just $rd\theta$. For a current circulating around the ring in the $\hat{\theta}$ direction, a vector length element pointing along the current direction is then $d\vec{l} = rd\theta\,\hat{\theta}$. We can now apply the Biot-Savart law:

$$\begin{aligned}
d\vec{B} &= \frac{\mu_o I}{4\pi}\frac{d\vec{l} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_o I}{4\pi}\frac{rd\theta\,\hat{\theta} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_o I}{4\pi r}d\theta\,\hat{\mathbf{z}} \\
\vec{B} &= \int\limits_{\text{circle}} d\vec{B} = \int\limits_0^{2\pi} \frac{\mu_o I}{4\pi r}d\theta\,\hat{\mathbf{z}} = \frac{\mu_o I}{4\pi r}d\theta\,\hat{\mathbf{z}}\,(2\pi) = \frac{\mu_o I}{2r}\,\hat{\mathbf{z}}
\end{aligned}$$

This is the field at the center of a *full* circle. Since the magnetic field obeys superposition, we could just as well say that our full circle is built out of two equivalent half circles, like the one above! The field from each half circle, by symmetry, must be half of the total field, so the field at the center of a semicircle must simply be

$$B = \frac{\mu_o I}{4r} \qquad \text{(semicircle, radius r)}$$

A more formal derivation goes just like the one above: simply replace the upper integration limit with π instead of 2π. Fundamentally, integrating the little dB's using the Biot-Savart law is just saying the field from any current distribution can be built out of the fields of infinitesimal line segments by superposition. That is what the integral is really "doing," it is building a circle out of tiny bits.

Anyway: for the problem at hand, we have two semicircular current segments contributing to the magnetic field at P: one of radius b, and one of radius a. The currents are in the opposite directions for the two loops, so their fields are in opposing directions. Based on the axes given, it is the outer loop of radius b that has its field pointing out of the page in the $\hat{\mathbf{z}}$ direction, and the inner loop of radius a in the $-\hat{\mathbf{z}}$ direction.

What about the straight bits of wire? The Biot-Savart law tells us that the magnetic field from a segment of the straight wire is proportional to $d\vec{l} \times \hat{\mathbf{r}}$. For the straight segments, $d\vec{l}$ and $\hat{\mathbf{r}}$ are parallel, and their cross product is zero. There is no field contribution at P from the straight segments! Thus, the total field is just that due to the semicircular bits,

$$\vec{B} = \frac{\mu_o I}{4b}\,\hat{\mathbf{z}} - \frac{\mu_o I}{4a}\,\hat{\mathbf{z}} = \frac{\mu_o I}{4}\left(\frac{1}{b} - \frac{1}{a}\right)\hat{\mathbf{z}}$$

4. Consider the magnetic field of a circular current ring, at points on the axis of the ring (use the exact formula, not your approximate form above). Calculate explicitly the line integral of the magnetic field along the ring axis from $-\infty$ to ∞, and check the general formula

$$\int \vec{B} \cdot d\vec{l} = \mu_o I_{\text{encl}} \tag{26}$$

Why may we ignore the "return" part of the path which would be necessary to complete a closed loop?

The magnetic field along the z axis of a current ring of radius R lying in the xy plane is

$$B_z = \frac{\mu_o IR^2}{2\left(z^2+R^2\right)^{3/2}} \tag{27}$$

If we take a path along z, then $d\vec{l} = d\hat{z}$

$$\int\limits_{-\infty}^{\infty} \vec{B}\cdot d\vec{l} = \int\limits_{-\infty}^{\infty} B_z\,dz = \int\limits_{-\infty}^{\infty} \frac{\mu_o IR^2}{2\left(z^2+R^2\right)^{3/2}}dz = \frac{\mu_o IR^2}{2}\left[\frac{z}{R^2\sqrt{z^2+R^2}}\right]_{\infty}^{\infty} = \mu_o I \tag{28}$$

Why does this work without a return path? Because at $z=\infty$ we can take a return path to $z=-\infty$ at an infinite distance from the ring, where the field is zero.

5. The electric field of a long, straight line of charge with λ coulombs per meter is

$$E = \frac{2k_e\lambda}{r}$$

where r is the distance from the wire. Suppose we move this line of charge parallel to itself at speed v. **(a)** The moving line of charge constitutes an electric current. What is the magnitude of this current? **(b)** What is the magnitude of the magnetic field produced by this current? **(c)** Show that the magnitude of the magnetic field is proportional to the magnitude of the electric field, and find the constant of proportionality.

The current can be found by thinking about how much charge passes through a given region of space per unit time. If we were standing next to the wire, in a time Δt, the length of wire that passes by us would be $v\Delta t$. The corresponding charge is then $\Delta q = \lambda v\Delta t$, and thus the current is

$$I = \frac{\Delta q}{\Delta t} = \frac{\lambda v\Delta t}{\Delta t} = \lambda v$$

From the current, we can easily find the magnetic field a distance r from the wire.

$$B = \frac{\mu_o I}{2\pi r} = \frac{\mu_o \lambda v}{2\pi r}$$

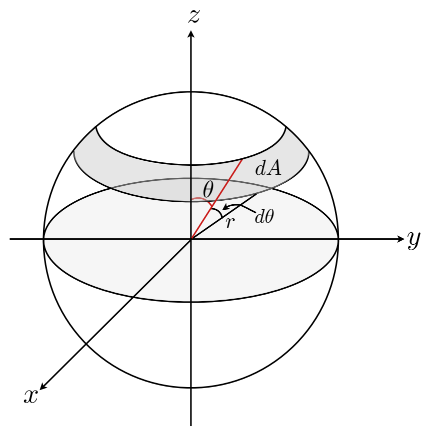
If the wire were sitting still (or we were traveling parallel to it at the same velocity v), it would produce the electric field given above. Rearranging the given expression, we can relate λ and E, $\lambda = Er/2k_e$. Substituting this in our expression for the magnetic field,

$$B = \frac{\mu_o\lambda v}{2\pi r} = \frac{\mu_o Erv}{4\pi k_e r} = \mu_o\epsilon_o vE = \frac{v}{c^2}E$$

For the last step, we noted that $\epsilon_o = 1/4\pi k_e$ and $c^2 = 1/\epsilon_o\mu_o$.

6. A sphere of radius R carries the charge Q which is distributed uniformly over the surface of the sphere with a density $\sigma = 4\pi R^2$. This shell of charge is rotating about an axis of the sphere with angular velocity ω, in radians/sec.

Find its magnetic moment. (Divide the sphere into narrow bands of rotating charge, find the current to which each band is equivalent and its dipole moment, and integrate over all bands.)



Start by dividing the surface into little strips, defined by a subtended angle $d\theta$ as in the figure above. The surface area of such a strip is its circumference $2\pi R\sin\theta$ times its width, the arclength R, $d\theta$:

$$da = 2\pi(R\sin\theta)(R\,d\theta)$$

The amount of charge on this strip can be found from the surface charge density $\sigma \equiv Q/4\pi R^2$:

$$dq = \sigma da = \frac{Q}{4\pi R^2}2\pi R^2\sin\theta\,d\theta = \frac{1}{2}Q\sin\theta\,d\theta \quad .$$

This bit of charge dq revolves at frequency $f = \omega/2\pi$, so it corresponds to a current $dI = dq/T = f\,dq$:

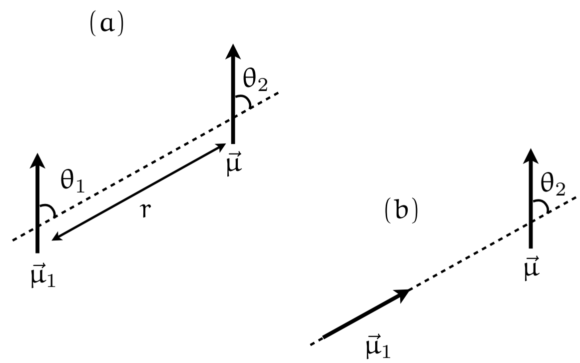
$$dI = f\,dq = \frac{\omega Q}{4\pi}\sin\theta\,d\theta$$

Each tiny current loop contributes a magnetic moment equal to its current times its area:

$$\mu = \frac{1}{c}\int A\,dI = \frac{2}{c}\int_0^{\pi/2}\pi(R\sin\theta)^2\frac{\omega Q}{4\pi}\sin\theta\,d\theta = \frac{\omega Q R^2}{2c}\int_0^{\pi/2}\sin^3\theta\,d\theta = \frac{\omega Q R^2}{3c}$$

7. We want to find the energy required to bring two dipoles from infinite separation into the configuration shown in (a) below, defined by the distance r apart and the angles θ_1 and θ_2. Both dipoles lie in the plane of the paper. Perhaps the simplest way to compute the energy is this: bring the dipoles in from infinity while keeping them in the orientation shown in (b). This takes no work, for the force on each dipole is zero. Now calculate the work done in rotating $\vec{\mu}_1$ into its final orientation while holding $\vec{\mu}_2$ fixed. Then calculate the work required to rotate $\vec{\mu}_2$ into its final orientation. Thus show that the total work done, which we may call the potential energy of the system, is

$$U = \frac{\mu_1\mu_2\left(\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2\right)}{r^3} \tag{29}$$



The potential of a single dipole in a magnetic field, up to an overall constant (which we can choose to be zero) is

$$U = -\vec{\mu} \cdot \vec{B}$$

Since we are worried about the net work done, the choice of zero for potential energy is irrelevant:

$$W = U_f - U_i$$

In the initial configuration, the field due to dipole μ_2 at dipole μ_1 is

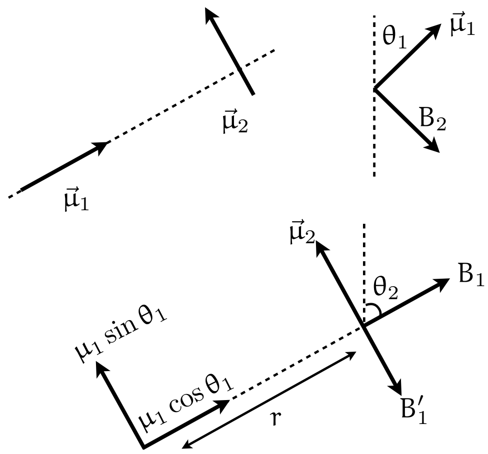
$$B_2 = \frac{\mu_2}{r^3}$$

The work required to rotate μ_1 from a $90°$ orientation to an angle θ_1 is thus

$$W_1 = U_f - U_i = -\mu_1 B_2 \cos(90+\theta_1) - 0 = \mu_1 B_2 \sin\theta_1 = \frac{\mu_1\mu_2}{r^3}\sin\theta_1$$

To next rotate μ_2, we break up the field from μ_1 into two parts, along the radial and angular parts (see problem 2; an infinitesimal current loop is essentially a magnetic dipole!)

$$B_1 = \frac{2\mu_1\cos\theta_1}{r^3} \qquad B_1' = \frac{\mu_1\sin\theta_1}{r^3} \quad .$$



The work to rotate μ_2 is then

$$\begin{aligned} W_2 &= U_f - U_i = [-\mu_2 B_1 \cos\theta_2 - \mu_2 B'_1 \cos(90+\theta_2)] - [-\mu_2 B'_1 \cos\pi] \\ &= -\mu_2 B_1 \cos\theta_2 + \mu_2 B'_1 \sin\theta_2 - \mu_2 B'_1 \\ &= -\frac{2\mu_1\mu_2}{r^3}\cos\theta_1\cos\theta_2 + \frac{\mu_1\mu_2}{r^3}\sin\theta_1\sin\theta_2 - \frac{\mu_1\mu_2}{r^3}\sin\theta_1 \quad . \end{aligned}$$

the total work is thus the sum of the work required to rotate both dipoles:

$$W = W_1 + W_2 = \frac{\mu_1\mu_2}{r^3}(\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2) \quad .$$

8. A metal crossbar of mass m slides without friction on two long parallel rails a distance b apart. A resistor R is connected across the rails at one end; compared with R, the resistance of the bar and rails is negligible. There is a uniform field $\vec{B}$ perpendicular to the plane of the figure. At time $t=0$, the crossbar is given a velocity v_o toward the right. What happens then? **(a)** Does the rod ever stop moving? If so, when? **(b)** How far does it go? **(c)** How about conservation of energy? *Hint: first find the acceleration, and make use of an instantaneous balance of power.*

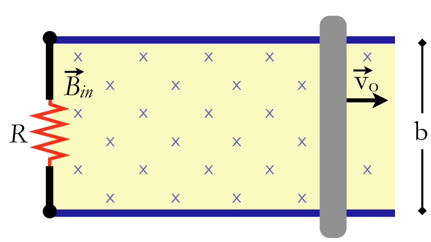


Figure 1: *Problem 5*

Let the position of the rod be x, with the $\hat{x}$ direction being to the right, and the $\hat{y}$ direction upward. This means the magnetic field points in the $-\hat{z}$ direction, giving in a magnitude $-B$. At time $t=0$, we will say the rod has velocity v_o and position x_o. For any time t, we will just call the velocity v and position x, since we don't know what they are yet.

As we discussed in class, the charges in the moving conducting bar will experience a magnetic force $F_b = qvB$. This will cause positive charges to experience a force in the $\hat{\mathbf{y}}$ direction, and negative charges in the $-\hat{\mathbf{y}}$ direction. Since charges are mobile in a conductor, this will cause positive charge to accumulate at the top of the rod, and negative charge at the bottom. This charge separation, over the width of the rod b, will lead to an electric field across the rod over the distance b. This electric field will serve to pull the charges back together.

Once an equilibrium is reached, the magnetic force separating the charges will be perfectly balanced by the electric force pulling them back together:

$$F_b = qvB = F_e = qE \qquad \Longrightarrow \qquad E = vB \tag{30}$$

Since electric field in a conductor is necessarily uniform, this implies a constant potential difference between the endpoints of the rod, given by $\Delta V = Eb$, with the top of the rod being at the higher potential (since positive charge accumulates there). Thus,

$$\Delta V = Bvb \tag{31}$$

The induced voltage - which is also applied to the resistor, since the rails are perfectly conducting - will lead to a counterclockwise current in the loop.

$$\Delta V = Bbv = IR$$

The presence of a current in the conducting rod will lead to a magnetic force. Since the field is into the page ($-\hat{\mathbf{z}}$ direction), and the current is flowing up through through the rod ($\hat{\mathbf{y}}$ direction), the force must be in the $\hat{\mathbf{y}}$ direction.

$$\vec{F}_B = I\vec{L} \times \vec{B} = IbB\,\hat{\mathbf{y}} \times (-\hat{\mathbf{z}}) = -IbB\,\hat{\mathbf{x}} = -\frac{B^2b^2v}{R} = ma$$

Recall that the direction of $\vec{L}$ is the same as the direction of the current. Since the magnetic force is the only force acting on the rod (in the absence of friction), it must also give the acceleration of the rod, as indicted in the last step. Incidentally, we could have gotten here much more quickly with a little intuition. If we recognize that there must be a current flowing in the resistor due to the induced voltage caused by the motion of the rod, then we know there is power dissipated in the resistor. This power must be the same as that supplied to the rod. The mechanical power is $\vec{F} \cdot \vec{v}$, and the electrical power is I^2R. Conservation of energy requires that these two powers be equal, which along with the motional voltage leads directly to the equation above.

Anyway: now we have a small equation relating v and its rate of change, $dv/dt = a$. We can solve it by separation of variables, which is totally cool since none of our quantities are zero. Dividing by zero is not cool.

$$-\frac{B^2b^2v}{R} = m\frac{dv}{dt}$$
$$\frac{mR}{B^2b^2}\frac{dv}{v} = -dt$$

Now we've got something we can integrate. Our starting condition is velocity v_o at time $t = 0$, going until some later time t where the velocity is v.

$$\begin{aligned}
\int_{v_o}^{v} \frac{mR}{B^2b^2}\frac{dv}{v} &= \int_0^t -dt \\
\frac{mR}{B^2b^2}\ln v\Big|_{v_o}^{v} &= -t\Big|_0^t \\
\frac{mR}{B^2b^2}\ln\frac{v}{v_o} &= -t \\
\implies \quad v = v_o e^{-t/\tau} &\qquad \text{with} \qquad \tau = \frac{mR}{B^2b^2}
\end{aligned}$$

The velocity is an exponentially decreasing function of time, which means it never stops moving - the velocity approaches, but does not reach, zero. The rod will also approach a final target displacement in spite of this fact, which we can find readily by integrating the velocity.

$$\Delta x = \int_o^\infty v\,dt = \int_o^\infty v_o e^{-t/\tau} = v_o\left(-\tau e^{-t/\tau}\right)\Big|_o^\infty = v_o\tau = \frac{mRv_o}{B^2b^2}$$

Once again, if you note that $1\,\text{T} = 1\,\text{kg/s}^2\cdot\text{A}$ and $1\,\text{V}\cdot 1\,\text{A} = 1\,\text{W}$, you should be able to make the units come out correctly.

Finally, we can calculate the total electrical energy expended. The electrical power dissipated in the resistor is $\mathscr{P} = dU/dt = I^2R$, so the tiny bit of potential energy dU expended in a time dt is $dU = I^2R\,dt$. We can integrate over all times to find the total potential energy.

$$\begin{aligned}
U &= \int_0^\infty I^2R\,dt = \int_0^\infty \left(\frac{Bbv}{R}\right)^2 R\,dt = \frac{B^2b^2}{R}\int_0^\infty v^2\,dt \\
&= \frac{B^2b^2v_o^2}{R}\int_0^\infty e^{-2t/\tau}\,dt = \frac{B^2b^2v_o^2}{R}\left(-\frac{\tau}{2}e^{-2t/\tau}\right)\Big|_0^\infty = \frac{B^2b^2v_o^2}{R}\left(\frac{mR}{2B^2b^2}\right) \\
&= \frac{1}{2}mv_0^2
\end{aligned}$$

As we would expect from conservation of energy, all of the initial kinetic energy of the conducting bar ends up dissipated in the resistor.