UNIVERSITY OF ALABAMA
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Problem Set 6: Solutions

1. *Ohanian 36.31* A flexible drive belt runs over two flywheels whose axles are mounted on a rigid base (Fig. 1). In the reference frame of the base, the horizontal portions of the belt have a speed v and therefore are subject to length contraction, which tightens the belt around the flywheels. However, in a reference frame moving to the right with the upper portion of the belt, the *base* is subject to length contraction, which ought to loosen the belt around the flywheels. Resolve this "paradox" with by a qualitative argument. *Hint: consider the lower portion of the belt as seen in the reference frame of the upper portion.*

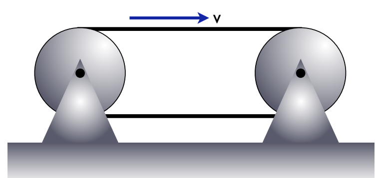

Figure 1: *Question 1*

Viewed from the laboratory frame, both the upper and lower belt should contract, as they are in motion relative to the observer. The fact that the top and bottom move in opposite directions does not matter in this case - both are contracted by the same amount, since length contraction depends on the *square* of the relative velocity. Thus, the belt appears to tighten.

Viewed from a frame traveling with the upper belt, the base appears to contract, since relative to the top portion of the belt, the pulleys on either side are moving away at velocity $|v|$. Why does the belt not loosen? This is because relative to the top of the belt, the bottom of the belt is moving at velocity $|2v|$ (ignoring the proper relativistic addition of velocities for the moment), and is thus length contracted twice as much as the the distance between the pulleys. Thus, the pulleys get closer together, but the bottom of the belt shortens even more, and overall the belt should appear to tighten.

Viewed from the bottom belt, the situation is reversed - both pulleys are moving at velocity v and the base contracts, but the top belt is moving at $|2v|$ and contracts twice as much. Still, the net effect is that the belt appears to tighten, a fact which all three reference frames agree on.

2. Show that the velocity of a relativistic particle can be expressed as follows:

$$\vec{v} = \frac{c\,\vec{p}}{\sqrt{m^2c^2 + p^2}}$$

The easiest way is to start with the right-hand side and show that it reduces to v. Since there is only one vector on either side, and the rest are only constants, we know that $\vec{v}$ and $\vec{p}$ must be in the same direction. Thus, it is sufficient to show that the magnitude of each side is the same, and we can drop the vector notation.

Start by substituting the relativistic expression for γ, and then multiply both numerator and denominator by c. After that, just start grouping terms ...

$$\begin{aligned}
\frac{cp}{\sqrt{m^2c^2 + p^2}} &= \frac{\gamma mvc}{\sqrt{m^2c^2 + p^2}} = \left(\frac{c}{c}\right)\frac{\gamma mvc}{\sqrt{m^2c^2 + p^2}} \\
&= \frac{\gamma mvc^2}{c\sqrt{m^2c^2 + p^2}} = \frac{\left(\gamma mc^2\right)v}{\sqrt{m^2c^4 + p^2c^2}} \qquad \left(\text{note } E = \gamma mc^2 \text{ and } E = \sqrt{m^2c^4 + p^2c^2}\right) \\
&= \frac{Ev}{E} = v
\end{aligned}$$

3. *Serway example 39.8* A car speeds past an observer on the ground at $0.9c$. A passenger in the car throws a ball out the car window at $0.7c$ relative to the car. What is the velocity of the ball with respect to the observer on the ground?

First, label what we know. Let the observer on the ground be in the unprimed frame, and the passenger in the car the primed frame:

$$\begin{aligned}
v_b &= \text{velocity of the ball relative to the ground} = ? \\
v_c &= \text{velocity of the car relative to the ground} = 0.9c \\
v'_b &= \text{velocity of the ball relative to the car} = 0.7c
\end{aligned}$$

Again, ask yourself how you would figure this out without relativity first, and that will help you pick the proper relativistic formula. Without relativity, you would just add the velocity of the car relative to the ground and the velocity of the ball relative to the car. Thus, all we need to do use our correct relativistic velocity addition formula:

$$v_b = \frac{v_c + v_b'}{1 + \frac{v_c v_b'}{c^2}}$$
$$= \frac{1.6c}{1 + (0.9)(0.7)} \approx 0.98c$$

4. *Leighton, "Principles of Modern Physics," 1.9* A cosmic-ray muon (μ) is moving vertically through the atmosphere with a speed $v = 0.99c$. Its mean life expectancy against radioactive decay into an electron and two neutrinos is 2.22 μs, as measured in its own "rest" system. What will be its mean life expectancy as viewed by an observer on earth?

The muon's lifetime is always the same in its own reference frame, a constant 2.22 μs which we will call the 'proper' time interval Δt_p. In the laboratory, we are in motion *relative to the muon*, and hence we measure a dilated (longer) time interval $\Delta t'$. For the given relative speed, then, we just need to calculate the dilated time interval, and that is the observed lifetime in the laboratory frame. The dilated interval is:

$$\Delta t' = \gamma \Delta t_p = \gamma (2.22\,\mu s)$$

Thus, we just need to calculate γ for the given speed v.

$$\Delta t_a' = \gamma (2.22\,\mu s) = \frac{1}{\sqrt{1 - (0.99c)^2 / c^2}} (2.22\,\mu s) = 15.7\,\mu s$$

5. *Leighton, 1.10* A stick of length L is at rest on one system and is oriented at an angle θ with respect to the x axis. What are the apparent length and orientation angle of this stick as viewed by an observer moving at a speed v with respect to the first system?

Let the reference frame at rest with respect to the stick be the 'unprimed' frame, with the primed frame corresponding to the observer moving at speed v relative to the stick. Since the relative motion is along the (presumed collinear) x and x' axes, the primed observer sees distances along the x' axis as contracted relative to the reference frame of the stick.

In the stick's (unprimed) frame, the horizontal extent of the stick along the x axis is $L_x = L\cos\theta$, while the extent along the y axis is $L_y = L\sin\theta$. For the moving observer, the x dimensions are contracted, but not the y, and thus

$$L'_x = L_x/\gamma = L_x\sqrt{1-\frac{v^2}{c^2}}$$
$$L'_y = L_y \tag{1}$$

The stationary observer sees the stick as having length $L = \sqrt{L_x^2 + L_y^2}$. The moving observer sees the stick as having a length

$$L' = \sqrt{(L'_x)^2 + \left(L'_y\right)^2} = \sqrt{L_x^2\left(1-\frac{v^2}{c^2}\right) + L_y^2} = L\sqrt{1-\frac{v^2}{c^2}\cos^2\theta} = L\sqrt{\frac{L_x^2}{\gamma^2} + L_y^2} = L\sqrt{\frac{\cos^2\theta}{\gamma^2} + \sin^2\theta} \tag{2}$$

6. *Leighton, 1.15* A particle appears to move with speed u at an angle θ with respect to the x axis in a certain system. At what speed and angle will this particle appear to move in a second system moving with speed v with respect to the first? Why does the answer differ from that of the previous problem?

It is most straightforward to assume that the two systems have their horizontal x axes aligned. This is still quite general, since we are still letting the particle move at an arbitrary angle θ, we may consider it to be a choice of axes and nothing more. Let the first frame, in which the particle moves with speed u at an angle θ be the 'unprimed' frame (x, y), and the second the 'primed' frame (x', y').

Along the x' direction in the primed frame, both perceived time and distance will be altered. Taking only the x' component of the velocity, we consider the particle's motion purely along the direction of relative motion of the two frames, and we may simply use our velocity addition formula. The x component of the particle's velocity will in the primed frame become

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} \tag{3}$$

Along the y' direction in the primed frame, since we consider motion of the particle orthogonal to the direction of relative motion of the frames, there is no length contraction. We need only consider time dilation. We derived this case in class, and the proper velocity addition for directions orthogonal to the relative motion leads to

$$u'_y = \frac{u_y}{\gamma\left(1 - u_x v/c^2\right)} \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{4}$$

The particle's speed in the primed frame is then easily calculated:

$$u' = \sqrt{u'_x u'_x + u'_y u'_y} = \left(\frac{u_x - v}{1 - u_x v/c^2}\right)^2 + \left(\frac{u_y}{\gamma\,(1 - u_x v/c^2)}\right)^2 \tag{5}$$

$$= \sqrt{\frac{(u_x - v)^2 + u_y^2/\gamma^2}{(1 - u_x v/c^2)^2}} = \frac{\sqrt{(u_x - v)^2 + u_y^2/\gamma^2}}{1 - u_x v/c^2} \tag{6}$$

As a double-check, we can set $\theta = 0$, such that $u_y = 0$, which corresponds to the particle moving along the x axis. Our expression then reduces to the usual one-dimensional velocity addition formula.

The direction of motion in the primed frame is also found readily:

$$\tan\theta' = \frac{u'_y}{u'_x} = \frac{u_y}{\gamma\,(1 - u_x v/c^2)} \frac{1 - u_x v/c^2}{u_x - v} = \left(\frac{u_y}{u_x - v}\right)\sqrt{1 - v^2/c^2} \tag{7}$$

7. The nonrelativistic expression for the momentum of a particle $p = mv$ agrees with experiments when $v \ll c$. For what speed does the nonrelativistic equation give an error of **(a)** 1.0%? **(b)** 5.0%?

First of all, what do we mean by error? You want to find percent error between momentum calculated with the relativistic formula (*viz.*, $|\vec{p}_{\text{rel}}| = \gamma m |\vec{v}|$) and the classical formula (*viz.*, $|\vec{p}_{\text{calss}} = m|\vec{v}|$). First, we will drop the vector notation now, since error in momentum will only be in magnitude, not direction. Let - $p_{\text{rel}} \equiv |\vec{p}_{\text{rel}}|$ and $p_{\text{class}} \equiv |\vec{p}_{\text{class}}|$. The definition of error you want is the difference between the two, divided by the correct one - the relativistic formula.

$$100\% \cdot \left|\frac{p_{\text{rel}} - p_{\text{class}}}{p_{\text{rel}}}\right| \leqslant \text{error desired}$$

For the last line, we drop the percent Now we can just plug in what we know:

$$\left|\frac{p_{\text{rel}} - p_{\text{class}}}{p_{\text{rel}}}\right| = \left|\frac{\gamma m v - m v}{\gamma m v}\right| = \left|\frac{\gamma \cancel{mv} - \cancel{mv}}{\gamma \cancel{mv}}\right| = \left|\frac{\gamma - 1}{\gamma}\right| \leqslant \text{error}$$

We can further simplify this:

$$\left|\frac{\gamma - 1}{\gamma}\right| = \left|1 - \frac{1}{\gamma}\right| \leqslant \text{error}$$
$$\left|1 - \text{error}\right| \leqslant \left|\frac{1}{\gamma}\right|$$

What we really want is v. Remember the equation for v in terms of γ from problem 2? Take that, and plug in the expression above:

$$v = c\sqrt{1-\frac{1}{\gamma^2}} \leqslant c\sqrt{1-\left|(1-\text{error})\right|^2}$$

Now all we need to do is plug in the desired minimum errors - 1% or 0.01 for **(a)**, and 5% or 0.05 for **(b)**:

$$\textbf{(a)} \quad v \leqslant c\sqrt{1-\left|(1-\text{error})\right|^2} = c\sqrt{1-\left|(1-0.01)\right|^2} \approx c\sqrt{0.02} \approx 0.14c$$

$$\textbf{(b)} \quad v \leqslant c\sqrt{1-\left|(1-\text{error})\right|^2} = c\sqrt{1-\left|(1-0.05)\right|^2} \approx c\sqrt{0.098} \approx 0.31c$$

8. An interstellar space probe is moving at a constant speed relative to earth of $0.76c$ toward a distant planet. Its radioisotope generators have enough energy to keep its data transmitter active continuously for 15 years, as measured in their own reference frame. **(a)** How long do the generators last as measured from earth? **(b)** How far is the probe from earth when the generators fail, as measured from earth? **(c)** How far is the probe from earth when the generators fail, *as measured by its built-in trip odometer*?

Just to be clear, we will label quantities measured in the earth's reference frame with primes ($'$), and quantities without primes are with respect to the probe's reference frame. The relative velocity between the earth and the probe is the same from both reference frames, $v = v'$. From the probe's (and its generators') reference frame, it is the observers on earth that are moving. The observers on earth should then see a *longer* time interval compared to the proper time measured on the probe:

$$\Delta t' = \gamma \Delta_p = \frac{15\,\text{yrs}}{\sqrt{1-\frac{(0.76c)^2}{c^2}}} \approx 23\,\text{yrs}$$

According to observers on earth, the generators should fail after a period of $\Delta t'$. Also according to them, the probe should have traveled a distance $d' = v'\Delta t'$ - the earth-bound observers watched the probe travel for an interval $\Delta t'$ at a constant velocity of v' in their reference frame:

$$d' = v'\Delta t' = (23\,\text{yrs})\left(3\times10^8\,\text{m/s}\right) \approx 2.2\times10^{17}\,\text{m}$$

Alternatively, we could express the distance in light years - the distance light travels in one year. To do that, we just have to realize that $0.76c$ means the probe travels at 76% of the speed of light:

$$d' = (0.76\,\text{light speed})\,(23\,\text{yrs}) \approx 18\,\text{light-years}$$

Finally, how about the distance traveled according to the probe? That is just the relative velocity multiplied by the elapsed time *from the probe's reference frame, i.e.*, the proper time:

$$d = v\Delta t = (15\,\text{yrs})\left(3\times10^8\,\text{m/s}\right)(0.76) = 1.1\times10^{17}\,\text{m} = 11\,\text{light-years}$$

9. *Ohanian 36.44* The acceleration of a particle in one reference frame is $a_x = dv_x/dt$, where the particle has an instantaneous velocity v_x in that frame. Consider a reference frame moving with speed V parallel to the positive x axis of the first frame. Show that the acceleration in the second frame is given by

$$a'_x = \frac{dv'_x}{dt'} = a_x \frac{\left(1 - V^2/c^2\right)^{3/2}}{\left(1 - v_x V/c^2\right)^3}$$

First thing: apply some calculus.

$$a'_x = \frac{dv'_x}{dt'} = \frac{dv'_x/dt}{dt'/dt} \tag{8}$$

What good is this? We know v'_x in terms of v_x and v, and we know t' in terms of t, so the two derivatives we need are trivial. Recall the velocity addition formula, applied to the current problem:

$$v'_x = \frac{v_x - v}{1 - v_x v/c^2} \tag{9}$$

We'll also need the Lorentz transformation for the time coordinates:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{10}$$

Here note that γ involves the relative velocity between the two reference frames, v, not the particle's velocity v_x. Thus, γ does not depend on t since v does not. Here x is just the current position of the particle in the unprimed frame; we won't need it since we're differentiating presently. Given these two transformations,

$$\frac{dt'}{dt} = \gamma\left(1 - \frac{vv_x}{c^2}\right) \tag{11}$$

$$\frac{dv'_x}{dt} = \frac{a_x - 0}{1 - vv_x/c^2} + \frac{-\left(v_x - v\right)\left(-a_x v/c^2\right)}{\left(1 - vv_x/c^2\right)^2} = a_x\left[\frac{1 - vv_x/c^2 + \left(v_x - v\right)\left(v/c^2\right)}{\left(1 - vv_x/c^2\right)^2}\right] \tag{12}$$

Thus,

$$a'_x = \frac{dv'_x}{dt'} = \frac{dv'_x/dt}{dt'/dt} = a_x \left[\frac{1 - vv_x/c^2 + (v_x - v)\left(v/c^2\right)}{\gamma\left(1 - vv_x/c^2\right)^3}\right]$$

$$= a_x \left[\frac{1 - v^2/c^2}{\gamma\left(1 - vv_x/c^2\right)^3}\right] = a_x \left[\frac{\left(1 - v^2/c^2\right)^{3/2}}{\left(1 - vv_x/c^2\right)^3}\right] \tag{13}$$

10. *Ohanian* A pion at rest ($m_\pi = 273\,m_{e^-}$) decays to a muon ($m_\mu = 207\,m_{e^-}$) and an antineutrino ($m_{\overline{\nu}} \approx 0$). This reaction is written as $\pi^- \rightarrow \mu^- + \overline{\nu}$. Find the kinetic energy of the muon and the energy of the antineutrino in electron volts. *Hint: relativistic momentum is conserved.*

Before the collision, we have only the pion, and since it is at rest, it has zero momentum and zero kinetic energy. After it decays, we have a muon and an antineutrino created and speed off in opposite directions (to conserve momentum). Both total energy - including rest energy - and momentum must be conserved before and after the collision.

First, conservation of momentum. Before the decay, since the pion is at rest, we have zero momentum. Therefore, afterward, the muon and antineutrino must have equal and opposite momenta. This means we can essentially treat this as a one-dimensional problem, and not bother with vectors. A consolation prize of sorts.

$$\text{initial momentum} = \text{final momentum} \tag{14}$$

$$p_\pi = p_\mu + p_\nu \tag{15}$$

$$0 = p_\mu + p_\nu \tag{16}$$

$$\Longrightarrow p_\nu = -p_\mu = -\gamma_\mu m_\mu v_\mu \tag{17}$$

For the last step, we made use of the fact that relativistic momentum is $p = \gamma m v$. Now we can also write down conservation of energy. Before the decay, we have only the rest energy of the pion. Afterward, we have the energy of both the muon and antineutrino. The muon has both kinetic energy and rest energy, and we can write its total kinetic energy in terms of γ and its rest mass, $E = \gamma m c^2$. The antineutrino has negligible mass, and therefore no kinetic energy, but we can still assign it a total energy based on its momentum, $E = pc$.

$$\text{initial energy} = \text{final energy} \tag{18}$$

$$E_\pi = E_\mu + E_\nu \tag{19}$$

$$m_\pi c^2 = \gamma_\mu m_\mu c^2 + p_\nu c \tag{20}$$

$$m_\pi = \gamma_\mu m_\mu + \frac{p_\nu}{c} \tag{21}$$

Now we can combine these two conservation results and try to solve for the velocity of the muon:

$$m_\pi = \gamma_\mu m_\mu + \frac{p_\nu}{c} = \gamma_\mu m_\mu - \gamma_\mu m_\mu \frac{v_\mu}{c} \tag{22}$$
$$\frac{m_\pi}{m_\mu} = \gamma_\mu - \gamma_\mu \frac{v_\mu}{c} = \gamma \left[1 - \frac{v_\mu}{c}\right] \tag{23}$$

We will need to massage this quite a bit more to solve for v_μ ...

$$\frac{m_\pi}{m_\mu} = \gamma \left[1 - \frac{v_\mu}{c}\right] = \frac{1 - \frac{v_\mu}{c}}{\sqrt{1 - \frac{v_\mu^2}{c^2}}} \tag{24}$$
$$\left(\frac{m_\pi}{m_\mu}\right)^2 = \frac{\left(1 - \frac{v_\mu}{c}\right)^2}{1 - \frac{v_\mu^2}{c^2}} = \frac{\left(1 - \frac{v_\mu}{c}\right)^2}{\left(1 - \frac{v_\mu}{c}\right)\left(1 + \frac{v_\mu}{c}\right)} = \frac{1 - \frac{v_\mu}{c}}{1 + \frac{v_\mu}{c}} \tag{25}$$

Now we're getting somewhere. Take what we have left, and solve it for v_μ ... we will leave that as an exercise to the reader, and quote only the result, using the given masses of the pion and muon:

$$\frac{v_\mu}{c} = \frac{1 - \left(\frac{m_\pi}{m_\mu}\right)^2}{1 + \left(\frac{m_\pi}{m_\mu}\right)^2} \approx -0.270 \tag{26}$$

From here, we are home free. We can calculate γ_μ and the muon's kinetic energy first. It is convenient to remember that the electron mass is $511\,\text{keV}/c^2$.

$$\gamma_\mu = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(0.27c)^2}{c^2}}} = \frac{1}{\sqrt{1 - 0.27^2}} \approx 1.0386 \tag{27}$$
$$KE_\mu = (\gamma_\mu - 1)\, m_\mu c^2 = (1.0386 - 1)\,(207 m_{e^-})\, c^2 \tag{28}$$
$$= 0.0386\left(207 \cdot 511\,\text{keV}/c^2\right) c^2 \approx 4.08 \times 10^6\,\text{eV} = 4.08\,\text{MeV} \tag{29}$$

Finally, we can calculate the energy of the antineutrino as well:

$$E_\nu = p_\nu c = -p_\mu c = -\gamma_\mu m_\mu v_\mu = -1.0386 \cdot \left(207 \cdot 5.11\,\text{keV}/c^2\right) \cdot (-0.270c) \tag{30}$$
$$\approx 2.96 \times 10^7\,\text{eV} = 29.6\,\text{MeV} \tag{31}$$