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PH 126 / LeClair

Fall 2009

Problem Set 7: Solutions

1. A thin ring of radius a carries a static charge q . This ring is in a magnetic field of strength B_0 , parallel to the ring's axis, and is supported so that it is free to rotate about that axis. If the field is switched off, how much angular momentum will be added to the ring? If the ring has mass m , show that it will acquire an angular velocity $\omega = qB_0/2mc$. *How is angular momentum being added? Consider the induced electric field and the resulting forces.*

It is not so clear, right off the bat, why the ring should start to rotate in the first place. Where can the angular momentum come from? The key is to recognize that time-varying magnetic field will give rise to a *circulating* electric field, which will cause a torque on the ring. One step at a time though!

At the instant the field is switched off, the magnetic flux through the loop changes from $\Phi_B = B_0\pi a^2$ to zero. The fact that the flux changes in time leads to an induced voltage around the ring, by Faraday's law:

$$\frac{\partial\Phi_B}{\partial t} = -\Delta V = \oint_{\text{ring}} \vec{E} \cdot d\vec{l}$$

If there is an induced voltage around the ring, then there must be an electric field as well, since the integral of $\vec{E} \cdot d\vec{l}$ around the loop is nonzero. What is the direction of the electric field? If the original B field was constant along the ring's axis, then the electric field resulting from its disappearance must be circulating. If the ring were conducting - and we did not say that it is - the change in magnetic flux would create induced currents circulating in such a way to stop the decrease in flux. A circulating current in the ring could only be caused by an electric field varying tangentially along the ring, i.e., a circulating electric field. The situation is the same whether the ring is conducting or not, in that the electric field will circulate around the axis of the ring.

The change in magnetic flux produces an electric field tangent to the ring everywhere. This tangential electric field is what will give rise to a torque. Each tiny element of the ring dl will feel an electric field E , and thus an electric force F_e in the tangential direction. Over the whole ring, adding up many tiny segments, there will be no net force, but there will be a net torque about the z axis.

The ring of radius a has a total charge q spread out uniformly. Let us define a linear charge density $\lambda = q/2\pi a$. Any tiny element dl of the ring then has a charge $dq = \lambda dl$. The electric field E will give rise to an electric force on each segment, also in the tangential direction:

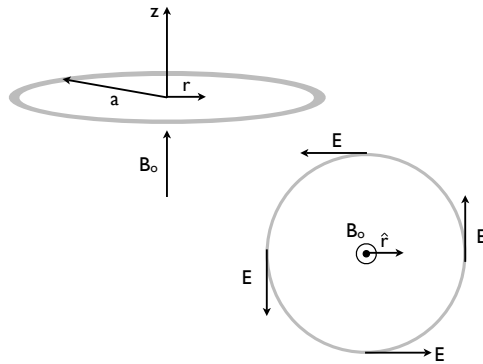


Figure 1: Problem 1: If one field is 'straight,' the other circulates.

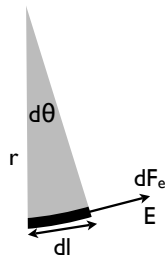


Figure 2: Problem 1: a tiny segment of the ring.

$$d\vec{F}_e = dq \vec{E} = \lambda dl E \hat{\theta}$$

On any little segment, we can now readily find the torque, since the electric force acts at a right angle to the radial direction. Remember that the ring is supported at its center point, which means the electric force acts over a distance a from the center of the circle to its rim.

$$d\vec{\tau} = \vec{a} \times d\vec{F}_e = a \hat{r} dl E \hat{\theta} = \lambda a dl E (\hat{r} \times \hat{\theta}) = \lambda a E dl \hat{z}$$

The torque is in the z direction, consistent with the right-hand rule. Once we know the torque on a small element dl , we can easily integrate to find the torque on the whole ring:

$$\vec{\tau} = \oint_{\text{ring}} d\vec{\tau} = \oint \lambda a E dl \hat{z}$$

Now we should notice two things: first, that the radius r is constant, and can be taken out of the integral. Second, since \vec{E} and $d\vec{l}$ are parallel everywhere, the integral of $E dl$ around the loop is nothing more than the potential difference around the ring ΔV !

$$\vec{\tau} = \lambda \alpha \oint \mathbf{E} \, d\mathbf{l} \hat{\mathbf{z}} = \lambda \alpha \Delta V \hat{\mathbf{z}} = \left(\frac{q}{2\pi\alpha} \right) \alpha \Delta V \hat{\mathbf{z}} = \left(\frac{q\Delta V}{2\pi} \right) \hat{\mathbf{z}}$$

The torque on the ring is simply proportional to the induced voltage, which is in turn proportional to the change in magnetic flux. We will drop the vector notation now, since we have the directions of everything straight at this point.

$$\tau = \left(\frac{q\Delta V}{2\pi} \right) = - \left(\frac{q}{2\pi} \right) \frac{d\Phi_B}{dt}$$

Of course, what is torque but the time rate of change of angular momentum ...

$$\frac{dL}{dt} = - \left(\frac{q}{2\pi} \right) \frac{d\Phi_B}{dt}$$

Now integrate both sides. Let the limits of integration be the state *i*, just before the field B_o is switched off, and the state *f* just after the field has been switched off.

$$\int_i^f dL = - \frac{q}{2\pi} \int_i^f d\Phi_B \quad \implies \quad L_f - L_i = \frac{q}{2\pi} (\Phi_{B,i} - \Phi_{B,f})$$

Just before the field is switched off, we have no net angular momentum, so $L_i = 0$, and just after the field is switched off, we have $\Phi_{B,f} = 0$. Since we know the initial magnetic flux as well,

$$L_f = \frac{q}{2\pi} (B_o \pi \alpha^2) = \frac{1}{2} q \alpha^2 B_o$$

So there is an angular momentum after the field is switched off, and its magnitude scales with the size of the initial field. In another way, we can say that the electric and magnetic fields also have angular momentum, just as they have an energy density. In fact, the fields carry energy, momentum, and angular momentum, a fact which is most easily seen from their effect on simple charge distributions like this one.

What about the rate of rotation? The net angular momentum for an object must equate to its moment of inertia I and rate of angular rotation ω : $\sum L = I\omega$. For a thin circular ring of mass m and radius α , we know that $I = m\alpha^2$.ⁱ This is the final thing we need:

$$L_f = I\omega = \omega m \alpha^2 = \frac{1}{2} q \alpha^2 B_o \quad \implies \quad \omega = \frac{q B_o}{2m}$$

In the end, ω depends only on the initial field strength, and the charge/mass ratio of the ring – not at all on the size of the ring.

ⁱOf course by “know” we mean “you can look these things up.”

2. An ocean current flows at a speed of 1.4 m/s in a region where the vertical component of the earth's magnetic field is 3.5×10^{-5} T. The resistivity of seawater in that region is about $\rho = 0.25 \Omega \text{ m}$. On the assumption that there is no other horizontal component of \vec{E} other than the motional term $\vec{v} \times \vec{B}$, find the horizontal current density J in A/m²? NB – recall the general version of Ohm's law, viz. $E = \rho J$. If you carried a bottle of seawater through the earth's field at this speed, would such a current be flowing in it?

Ions in the seawater in motion experience a magnetic force qvB , which will separate positive and negative ions. This results in an electric force qE . In equilibrium, the two will balance, giving $E = vB$. Using Ohm's law, $J = E/\rho = vB/\rho$.

More formally, let the ocean current be along the \hat{y} direction and the magnetic field along the \hat{z} direction. The electric field in the frame of the moving ions is then

$$\vec{E} = \vec{v} \times \vec{B} = vB \hat{x} \quad (1)$$

The current density is then given by Ohm's law:

$$\vec{j} = \frac{1}{\rho} \vec{E} = \frac{vB}{\rho} \hat{x} \approx 1.96 \times 10^{-4} \text{ A/m}^2 \quad (2)$$

3. A taut wire passes through the gap of a small magnet, where the field strength is 0.5 T. The length of the wire within the gap is 1.8 cm. Calculate the amplitude of the induced alternating voltage when the wire is vibrating at its fundamental frequency of 2000 Hz with an amplitude of 0.03 cm.

We know that a vibrating string can be described by a sine wave, and at its fundamental mode, the string has the shape of one-half cycle of a sine wave – it is pinned at two end points, and gradually reaches a maximum deflection halfway between the ends. In this case, the amplitude of the vibration $A = 3 \times 10^{-4}$ m is very small compared to the length of the wire in the gap, $l = 0.018$ m. Even if the wire were exactly 0.018 m long and no longer, the deflection of the wire would still only be about 2%. The curvature of the wire inside the magnet is *tiny* compared to even its minimum length, which means we may treat the wire as still essentially straight, rather than a sine wave. As the wire vibrates, we will treat the segment within the magnet as simply a straight segment which oscillates up and down, ignoring the slight curvature.

A given point on a vibrating wire can be described by $y = A \cos(\omega t)$, where y is the height of the string above or below its mean position, A is the amplitude of vibration, and $\omega = 2\pi f_0$ is the frequency of vibration. The velocity of the wire at a given point is easily found:

$$v_y = \frac{dy}{dt} = \frac{d}{dt} [A \cos(2\pi f_0 t)] = -2\pi f_0 A \sin(2\pi f_0 t)$$

If we treat the portion of the vibrating wire inside the magnet as still essentially straight, what we have is a straight segment of wire of length l moving at velocity v_y perpendicularly to a magnetic field B . This means we must have an induced voltage, as we do any time we have a conductor moving in a magnetic field. Presume that the wire is oscillating in a direction perpendicular to the magnetic field. In that case, the induced voltage on our "straight" segment vibrating up and down is

$$\Delta V = -Blv_y = 2\pi f_o B l A \sin(2\pi f_o t)$$

So, indeed there is an induced alternating voltage, which has the same frequency as the vibrational frequency of the wire (though it has a $\pi/2$ phase shift). We are asked to find only the amplitude of the induced voltage, which means just the pre-factor of the sine term above:

$$|\Delta V| = 2\pi f_o B l A = (2\pi) (2 \times 10^3 \text{ s}^{-1}) (0.5 \text{ T}) (1.8 \times 10^{-2} \text{ m}) (3 \times 10^{-4} \text{ m}) \approx 0.034 \text{ V}$$

In order for the units to work out correctly, remember that $1 \text{ T} = 1 \text{ V} \cdot \text{s} / \text{m}^2$.

4. Derive an approximate formula for the mutual inductance of two circular rings of the same radius a , arranged like wheels on the same axis with their centers a distance b apart. Use an approximation good for $b \gg a$. Recall that mutual inductance is the ratio of the flux through one object (say, ring 1) per unit current in the other object (say, ring 2).

With a current I in ring 1, the field at a distance b along ring 1's axis is

$$B_1 = \frac{\mu_o I}{2} \frac{a^2}{(a^2 + b^2)^{3/2}} \quad (3)$$

We derived this in class as a special case of a problem from HW 5. For $b \gg a$, this can be approximated

$$B_1 = \frac{\mu_o I}{2} \frac{a^2}{(a^2 + b^2)^{3/2}} = \frac{\mu_o I a^2}{2b^3} \frac{1}{(1 + a^2/b^2)^{3/2}} \approx \frac{\mu_o I a^2}{2b^3} \quad (4)$$

Further, if we are far from ring 2 ($b \gg a$), we can assume that the field due to ring 1 is reasonably uniform over the area of ring 2. That is, we take B from ring 1 to be constant over loop 2, and equal to its value at the center of loop 2. The flux through ring 2 is then approximately

$$\Phi_{12} = B_1 \pi a^2 \approx \frac{\mu_o \pi I a^4}{2b^3} \quad (5)$$

The mutual inductance is the flux through ring 2 divided by the current in ring 1 causing it:

$$M \approx \frac{\mu_0 \pi a^4}{2b^3} \quad (6)$$

5. A conducting bar of length l rotates with a constant angular speed ω about a pivot at one end. A uniform magnetic field \vec{B} is directed perpendicular to the plane of rotation, as shown in the figure below. Find the voltage induced between the ends of the bar.

Here is the situation we are given:

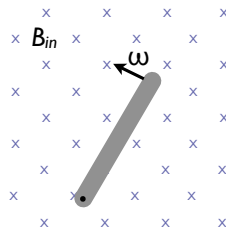


Figure 3: Problem 5: a conducting bar, rotating about a pivot on one end.

Take a tiny slice of the rod of thickness dl a distance r from the pivot point:



Figure 4: Taking a small slice of the rod

This particular slice, a distance r from the pivot, is moving at linear velocity $v = r\omega$. The force on the charges in this tiny slice of the rod, moving perpendicular to the external magnetic field, is $F = qvB = qr\omega B$ pointing toward the pivot point for positive charges. This force serves to separate positive and negative charges within the slice – positive charge accumulates on the side closed to the pivot point, negative on the far side. This charge separation over the length of the slice dl leads to an electric field and thus an electric potential over the length of the slice. In equilibrium the two forces will balance:

$$F_b = qvB = qr\omega B = qE \quad (7)$$

We could also just consider transforming the magnetic field in the external reference frame to the reference frame of the charges in the rod, where the field will be purely electric:

$$\vec{E} = \vec{v} \times \vec{B} \quad \implies \quad E = vB = r\omega B \quad (8)$$

The electric field in the rod will be uniform, and thus the electric potential dV across our tiny slice of the rod is given by $\vec{E} = -\vec{\nabla} V$, or simply

$$dV = E dl \quad (9)$$

The potential between the endpoints of the rod is found by integrating over all such tiny slices, or if you like, integrating $\vec{E} \cdot d\vec{l}$ over the length of the rod. Let the potential difference be defined as the potential of the tip minus the potential of the pivot point.

$$\Delta V = - \int_0^l E dl = - \int_0^l r\omega B dl = -\frac{1}{2}\omega B l^2 \quad (10)$$

The minus sign reminds us that the pivot point of the rod is positive, while the tip is negative, consistent with the force on the charges in the rod and the resulting charge accumulation.

6. A capacitor consists of two parallel rectangular plates with a vertical separation of 0.02 m. The east-west dimension of the plates is 0.2 m, the north-south dimension is 10 cm. The capacitor has been charged by connecting it temporarily to a battery of 300 V.

- (a) How many excess electrons are on the negative plate?
- (b) What is the electric field strength between the plates?

Now, give the quantities as they would be measured in a frame of reference which is moving eastward, relative to the laboratory in which the plates are at rest, with speed $0.6c$.

- (c) The dimensions of the capacitor,
- (d) The number of excess electrons on the negative plate,
- (e) The electric field strength between the plates.

Now answer the same questions above (c-e) for a frame of reference which is moving upward with speed $0.6c$.

- (a) The excess charge can be found from the definition of the capacitance and its specific form for two

parallel plates:

$$C = \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{d} \quad \Longrightarrow \quad Q = \frac{\epsilon_0 A \Delta V}{d} \approx 2.665^{-9} C \approx 1.66 \times 10^{10} \text{ electrons} \quad (11)$$

Here ΔV is the potential difference applied to the battery, the area of the plates is the product of the east-west and north-south distance, $l_{ew}l_{ns}$, and d is the vertical separation.

(b) The electric field strength between the plates, treating them as infinite plates, can be found in two ways:

$$E = \frac{\Delta V}{d} = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A} = 15,000 \text{ V/m} \quad (12)$$

(c) Moving eastward, perpendicular to the direction separating the plates, we will have a contraction of the east-west length but *not* the north-south length or the separation. Thus, the new dimensions of the capacitor are

$$l'_{ew} = l_{ew}/\gamma = l_{ew}\sqrt{1 - v^2/c^2} = 0.16 \text{ m} \quad (13)$$

$$l'_{ns} = l_{ns} = 0.1 \text{ m} \quad (14)$$

$$d' = d = 0.02 \text{ m} \quad (15)$$

(d) The number of electrons per plate is the same, since charge is invariant.

(e) The electric field strength will increase, since we have the same number of electrons confined to effectively smaller plates. The area of the plates is now a factor of gamma smaller, $l'_{ew}l'_{ns} = l_{ew}l_{ns}/\gamma$, meaning the charge density is a factor γ higher, and thus the electric field is also a factor of γ higher:

$$E' = \frac{\sigma'}{\epsilon_0} = \frac{\gamma\sigma}{\epsilon_0} = \gamma E = 18,750 \text{ V/m} \quad (16)$$

When the motion of the capacitor is upward, only the distance between the plates is contracted:

$$l'_{ew} = l_{ew} = 0.2 \text{ m} \quad (17)$$

$$l'_{ns} = l_{ns} = 0.1 \text{ m} \quad (18)$$

$$d' = d/\gamma = 0.016 \text{ m} \quad (19)$$

The number of electrons remains the same, as does the charge density in this case. The plates move closer together, but for an infinite parallel plate capacitor, the electric field *does not depend on the plate spacing*,

$E = \sigma/\epsilon_0$. Thus, the electric field is unchanged when the motion is along the direction of the electric field.

7. Consider the trajectory of a charged particle which is moving with speed $0.8c$ in the x direction when it enters a large region in which there is a uniform electric field in the y direction. Show that the x velocity of the particle must actually *decrease*. What about the x component of momentum?

The electric field applies a force qE , but only in the y direction. This means that the x component of the momentum must be conserved:

$$0 = F_x = \frac{dp_x}{dt} \quad \implies \quad p_x \text{ is conserved} \quad (20)$$

So that's that then, right? Not quite. Momentum involves both distance and time, so we have both time dilation and length contraction to worry about. Momentum is conserved in the x direction, but we must use *relativistic* momentum. Let i subscripts represent quantities before the particle enters the region of electric field, and f subscripts those after the particle is in the electric field. Comparing the two momenta,

$$p_{xi} = \gamma_i m v_{xi} = p_{xf} = \gamma_f m v_{xf} \quad (21)$$

Here the γ factors are

$$\gamma_i = \frac{1}{\sqrt{1 - \frac{v_{xi}^2}{c^2}}} \quad \gamma_f = \frac{1}{\sqrt{1 - \frac{v_{xf}^2 + v_{yf}^2}{c^2}}} \quad (22)$$

This implies

$$\frac{v_{xi}}{\sqrt{1 - \frac{v_{xi}^2}{c^2}}} = \frac{v_{xf}}{\sqrt{1 - \frac{v_{xf}^2 + v_{yf}^2}{c^2}}} \quad (23)$$

$$\implies \quad v_{xf} = v_{xi} \sqrt{1 - \frac{v_{yf}^2}{c^2}} \quad (24)$$

Since there *is* a force acting in the y direction, we know that the particle will acquire a velocity in the y direction, and $v_{xf} < v_{xi}$. The x -component of the velocity, and thus the x -component of the momentum must decrease.

Note that for non-relativistic speeds, $v \ll c$, we recover the Newtonian result, viz., the x -component of the velocity remains constant.

8. Very large magnetic fields can be produced using a procedure called *flux compression*. A metallic cylindrical tube of radius R is placed coaxially in a long solenoid of somewhat larger radius. The space between the tube and the solenoid is filled with a highly explosive material. When the explosive is set off, it collapses the tube to a cylinder of radius $r < R$. If the collapse happens very rapidly, induced current in the tube maintains the magnetic flux nearly constant inside the tube, even though the area shrinks. If the initial magnetic field in the solenoid is 2.50 T, and $R/r = 12.0$, what is the maximum field that can be reached?

The basic idea here is that the *flux* through the tube is the same before and after the explosion. Since after the explosion the cross-sectional area is severely reduced, the field must be much larger in order to make the flux the same. Here's a bit from the Wikipedia about flux compression, explaining things in more detail:

Magneto-explosive generators use a technique called "magnetic flux compression", which will be described in detail later. The technique is made possible when the time scales over which the device operates are sufficiently brief that resistive current loss is negligible, and the magnetic flux on any surface surrounded by a conductor (copper wire, for example) remains constant, even though the size and shape of the surface may change.

This flux conservation can be demonstrated from Maxwell's equations. The most intuitive explanation of this conservation of enclosed flux follows from the principle that any change in an electromagnetic system provokes an effect in order to oppose the change. For this reason, reducing the area of the surface enclosed by a conductor, which would reduce the magnetic flux, results in the induction of current in the electrical conductor, which tends to return the enclosed flux to its original value. In magneto-explosive generators, this phenomenon is obtained by various techniques which depend on powerful explosives. The compression process allows the chemical energy of the explosives to be (partially) transformed into the energy of an intense magnetic field surrounded by a correspondingly large electric current.

– Wikipedia, "Flux Compression"

So: all we need to do is calculate the flux before and after the explosion, set them equal to each other, and solve for the field after the explosion. Quantities with 'i' subscripts refer to before the explosion, those with 'f' after the explosion, and all symbols have their usual meanings.

$$\begin{aligned}\Phi_{B,i} &= B_i A_i = B_i \cdot \pi R^2 \\ \Phi_{B,f} &= B_f A_f = B_f \cdot \pi r^2 \\ \Phi_{B,i} &= \Phi_{B,f} \\ \implies B_i \pi R^2 &= B_f \pi r^2 \\ B_f &= \left(\frac{R}{r}\right)^2 B_i = (12.0)^2 \cdot 2.50 \text{ T} = 360 \text{ T}\end{aligned}$$