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Jones calculus

1 Rotation of coordinate systems

1.1 Two dimensions

Take a normal (x, y) system, and pick a point $P(x, y)$. A *counterclockwise* rotation about the origin by θ (rotating x toward y in a right-handed system) means that the point $P(x, y)$ is described by the coordinates $P'(x', y')$ in the rotated frame, as shown below.

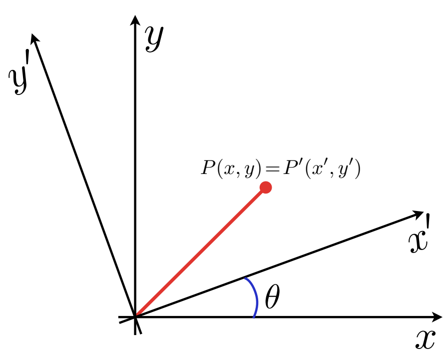

Figure 1: *Rotation of a 2D coordinate system.*

Using basic trigonometry, we can readily find a relationship between the two coordinate representations:

$$x' = x\cos\theta - y\sin\theta \qquad x = x'\cos\theta + y'\sin\theta \tag{1}$$

$$y' = x\sin\theta + y\cos\theta \qquad y = x'\sin\theta - y'\cos\theta \tag{2}$$

Or, writing this as a matrix equation,

$$P'_\theta = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} P \equiv R(\theta)P \tag{3}$$

$$P = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \equiv R'(\theta)P'_\theta \tag{4}$$

Here, the matrix $R(\theta)$ acting on the column vector P is known as a *rotation matrix*, quite sensibly. Rotation matrices must have a determinant of ± 1 in order to preserve length.[i]

On the other hand, if make a *clockwise* rotation of θ about the origin (rotating y toward x in a right-handed system), the new coordinates after rotation are

$$x' = x\cos\theta + y\sin\theta \tag{5}$$

$$y' = -x\sin\theta + y\cos\theta \tag{6}$$

In terms of the rotation matrix,

$$P'_{-\theta} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} P \equiv R(-\theta)P \tag{7}$$

$$\tag{8}$$

Note that as expected $R(\theta)$ and $R(-\theta)$ are the same if one substitutes $-\theta$ for θ, and vice versa.[ii]

This much follows from basic trigonometry. Below are a few examples of specific rotations in two dimensions.

$$R(90^\circ) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{Counterclockwise by } 90^\circ$$

$$R(180^\circ) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2 \qquad 180^\circ;\ I_2 = \text{identity matrix}$$

$$R(-90^\circ) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -R(90^\circ) \qquad \text{Clockwise by } 90^\circ \tag{9}$$

Note that $R(180^\circ) = R(-90^\circ)R(-90^\circ) = R(90^\circ)R(90^\circ)$, or that two successive 90° rotations is the same as a 180° rotation (you can think of successive multiplications by rotation matrices as suc-

[i] As a result, rotations also preserve the angles between vectors.
[ii] One can equivalently say that $R(\theta)$ is the matrix transpose of $R(-\theta)$.

cessive rotations). Further, $R(-90°)R(90°) = I_2$ and $R(-90°)R(90°) = I_2$ give no net rotation, and one obtains the identity matrix I_2 as required.

Another point to note is that for a rotation about a fixed origin, distances from the origin must be preserved. This requires, in two dimensions,

$$x^2 + y^2 = x'^2 + y'^2$$

which you can verify is true for relationships above. This is equivalent to requiring that the rotation matrices have a determinant of ± 1 ($\det R = \pm 1$), where $+$ and $-$ correspond to proper and improper rotations, respectively (see below). This also makes the rotation matrices *orthogonal transformations*.

1.2 Three dimensions

In three dimensions, we have three types of rotations about a fixed origin to consider: rotation of the x, y, and z axes in a counterclockwise direction while looking at the origin. The three basic rotation matrices (for a right-handed coordinate system) are

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad \text{rotates } y \text{ axis toward } z \text{ axis} \tag{10}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & 1 \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \text{rotates } z \text{ axis toward } x \text{ axis} \tag{11}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{rotates } x \text{ axis toward } y \text{ axis} \tag{12}$$

Any rotation can be given as a composition of rotations about these three axis, and can thus be represented by a 3×3 matrix operating on a column vector describing position.

1.3 Proper and Improper Rotations: Handedness

We have stated already that we always choose a *right-handed* coordinate system. What this means is that if in two dimensions we choose an x axis, have two possible choices for an orthogonal y axis (and in 3D, picking x and y leaves two choices for z). The two possible choices are left-handed, and right-handed. In the two-dimensional right-handed case, if your right index finger points along x, your right thumb points along y (or, if x increases to the right, then y increases upward). This is in contrast to a *left-handed* system, in which y would run downward if x ran to the right. Left-handed and right-handed systems cannot be interchanged by a pure rotation (convince yourself of this!), but require an *inversion* in conjunction with a rotation, or a *mirror plane*.

Inversion means changing the signs of all of the coordinates, e.g., $(x,y,z)\rightarrow(-x,-y,-z)$. Inversion followed by rotation is called an *improper rotation*. An improper rotation of an object produces a rotation of its mirror image, and thus changes left-handed to right-handed or vice versa. A mirror plane means switching the sign of only one of the coordinates, e.g., $(x,y,z)\rightarrow(x,y,-z)$, which also changes left-handed into right-handed. Inspection of the rotation matrices is enough to tell you whether you are dealing with a proper or improper rotation. An (orthogonal matrix) A is classified as proper (corresponding to pure rotation) if $\det(A)=1$, where $\det(A)$ is the determinant of A, or improper if $\det(A)=-1$. For example, using the rotation matrix in 2D given above,

$$\det R = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \tag{13}$$

On the other hand, the following matrix must be an improper rotation:

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{14}$$

$$\det A = \begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{vmatrix} = -\cos^2\theta - \sin^2\theta = -1 \tag{15}$$

For the special case $\theta=0$, you can see that this matrix carries (x,y) into $(x,-y)$, which can also be viewed as a mirror operation (reflecting y about a mirror placed along the x axis). For $\theta=90°$, this matrix swaps the x and y axis (reflection about $y=x$).

2 Jones vectors

Consider a plane electromagnetic wave traveling along the z direction, with electric field components along x and y. The electric field is then $\vec{E}(z,y)=E_x(z,t)\,\hat{\imath}+E_y(z,t)\,\hat{\jmath}$, and the field components can be written, with arbitrary phases φ_x and φ_y,

$$E_x(z,t) = E_{0x}e^{i(kz-\omega t+\varphi_x)} \tag{16}$$

$$E_y(z,t) = E_{0y}e^{i(kz-\omega t+\varphi_y)} \tag{17}$$

Here E_{0x} and E_{0y} are the purely real maximum amplitudes. The overall components are in general still complex, owing to the complex exponential propagator and phases. The propagator $\omega t - kz$ is common to both components, and may be suppressed if we only wish to calculate relative field amplitudes:

$$E_x = E_{0x} e^{i\varphi_x} \tag{18}$$
$$E_y = E_{0y} e^{i\varphi_y} \tag{19}$$

We can write the field more compactly as a 2×1 matrix:

$$\vec{E} = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{bmatrix} \tag{20}$$

This column matrix is called the Jones column matrix, or simply the Jones vector. In this general form, where the phases and maximum amplitudes for x and y components are arbitrary, we have elliptically polarized light. We can add these Jones vectors together just as we added electric field vectors together to find the total field.

The intensity I of the optical field is found from the squared magnitude of the electric field:[iii]

$$I = E_x E_x^* + E_y E_y^* = |E_x|^2 + |E_y|^2 \tag{21}$$

where $*$ denotes complex conjugation (since the amplitudes are in general still complex, owing to the phase factors). Using matrix multiplication, the equivalent operation is to multiply each Jones vector by its *complex transpose*. For a column vector, the complex transpose is not hard to find:

$$\vec{E}^\dagger = \begin{bmatrix} E_x^* & E_y^* \end{bmatrix} \tag{22}$$

The intensity in matrix notation is then

$$I = \vec{E}^\dagger \vec{E} = \begin{bmatrix} E_x^* & E_y^* \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} E_{0x} e^{-i\varphi_x} & E_{0y} e^{-i\varphi_y} \end{bmatrix} \begin{bmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{bmatrix} = E_{0x}^2 + E_{0y}^2 \equiv E_0^2 \tag{23}$$

Most of the time, we define $E_0^2 = 0$ for convenience, and deal with normalized intensities.

[iii]Modulo an overall constant depending on the system of units chosen.

The Jones formalism has the advantage of being compact, relatively simple, and much like the matrix algebra you will learn for handling spin in quantum mechanics. Its disadvantages are that it can *only* handle polarized light, and use of complex matrices can be at times tedious and unwieldy. Typically, the Jones formalism is used when adding *amplitudes* is necessary, whereas the Mueller formalism (which can handle unpolarized light) is most often used when one needs only to add intensities.

3 Jones vectors for various polarization states

The simplest type of polarization is *plane polarized* light, or $\mathscr{P}$, characterized by the x and y components being perfectly in- or out-of-phase ($\delta_y - \delta_x = n\pi$). First consider horizontal linearly polarized light. In this case, $E_y = 0$, and

$$\vec{E} = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ 0 \end{bmatrix} \tag{24}$$

Since we have only one component, the phase factor is irrelevant, and we may suppress it. Further, it is most convenient to normalize the incident intensity $E_{0x}^2 = 1$, so we may write horizontal linearly polarized light as

$$\vec{E} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{25}$$

By analogy, vertical linearly polarized light is

$$\vec{E} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{26}$$

Right-handed circularly polarized light, or $\mathscr{R}$, is characterized by $\varphi_y - \varphi_x = -\frac{\pi}{2}$, $E_{0y} = E_{0x}$. This gives $2E_{0x}^2 = 1$, and noting $e^{i\pi/2} = i$:

$$\vec{E} = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = E_0 e^{i\varphi_x} \begin{bmatrix} 1 \\ e^{i(\varphi_y - \varphi_x)} \end{bmatrix} = E_0 e^{i\varphi_x} \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{27}$$

Suppressing the phase factor, and using the normalization condition,

$$\vec{E} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{28}$$

A similar expression holds for left-handed circularly polarized light, or $\mathscr{L}$; the most general form for elliptically-polarized light is already given above. Below, we summarize the Jones vectors for common polarization states.

$$\text{horizontal } \mathscr{P} \text{ polarization: } \mathsf{H} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{29}$$

$$\text{vertical } \mathscr{P} \text{ polarization: } \mathsf{V} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{30}$$

$$\mathscr{P} \text{ polarization } 45° \text{ from } x \text{ axis:} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{31}$$

$$\mathscr{P} \text{ polarization } -45° \text{ from } x \text{ axis:} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{32}$$

$$\mathscr{R} \text{ circular polarization } \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{33}$$

$$\mathscr{L} \text{ circular polarization } \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{34}$$

We can note a few interesting facts here. First, the basic polarization states come in *orthonormal pairs*. Two vectors $\vec{A}$ and $\vec{B}$ are said to be *orthogonal* if their dot product is zero. In complex matrix notation, $\vec{A}^\dagger \vec{B} = 0$. If this condition is satisfied and both vectors are normalized to unit length, they are said to be *orthonormal* (for example, the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ are orthonormal). This is true for our linearly horizontal and vertically polarized light:

$$\begin{bmatrix} 1 & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \tag{35}$$

It is also true for right- and left-circularly polarized light:

$$\begin{bmatrix} 1 & i \end{bmatrix}^\dagger \begin{bmatrix} 1 \\ -i \end{bmatrix} = 0 \tag{36}$$

Thus, the horizontal and vertical states form an orthonormal basis, from which we can construct any other polarization, and the same is true for the right- and left-handed circular polarizations. For example, let us ad horizontal and vertically polarized light of different amplitudes and phases:

$$\vec{E} = \vec{E}_H + \vec{E}_V = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} \tag{37}$$

This is the Jones vector for elliptically-polarized light, demonstrating that it can be created by linear combination of orthogonal linearly polarized sources. If we make the restrictions that the x and y amplitudes are the same, insist on a $\pi/2$ phase shift, $E_{0x} = E_{0y} \equiv E_0$, $\varphi_y - \varphi_x = \pm\pi/2$, we generate circularly polarized light. On the other hand, if we make the restrictions that the x and y amplitudes and phases are the same, $E_{0x} = E_{0y} \equiv E_0$, $\varphi_y = \varphi_x \equiv \varphi$, we obtain linear polarization rotated $+45°$ from the x axis:

$$\vec{E} = \vec{E}_H + \vec{E}_V = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} E_0e^{i\varphi} \\ E_0e^{i\varphi} \end{bmatrix} E_0e^{\varphi} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{38}$$

The latter result can be more simply derived by simply adding normalized horizontal and vertical linear polarizations without phase offsets:

$$\vec{E} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{39}$$

Apart from the overall normalization factor, the result is the same. Thus, from our orthonormal basis of horizontal and vertical polarization (henceforth H and V), we can generate any other polarization state. We can do the same with a basis of right- and left-circularly polarized light (henceforth $\mathscr{R}$ and $\mathscr{L}$). Consider adding $\mathscr{R}$ and $\mathscr{L}$ with equal amplitudes:

$$\vec{E}_t = \vec{E}_L + \vec{E}_R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{40}$$

Apart from normalization, we have generated horizontally polarized light from circularly polarized light! We can generate vertically polarized light by adding a π phase shift to one of the incident waves:

$$\vec{E}_t = \vec{E}_L + \vec{E}_R = \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{\sqrt{2}}{i} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{41}$$

Here the overall factor of i is just a π phase shift for both components, and may be suppressed. You can convince yourself easily enough that if you add $\mathscr{R}$ and $\mathscr{L}$ with *unequal* amplitudes, elliptical polarization is generated.

4 Jones matrices for various components

$$\text{horizontal linear polarizer: } M_{\leftrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{42}$$

$$\text{vertical linear polarizer: } M_{\updownarrow} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{43}$$

$$\text{linear polarizer } 45^\circ \text{ from horizontal:} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{44}$$

$$\text{linear polarizer } -45^\circ \text{ from horizontal:} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{45}$$

$$\mathscr{R} \text{ circular polarizer:} \frac{1}{2} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \tag{46}$$

$$\mathscr{L} \text{ circular polarizer:} \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \tag{47}$$

$$\text{linear polarizer at angle } \theta \text{ from horizontal: } M_{\mathscr{P},\theta} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \tag{48}$$

$$\text{quarter wave plate, fast axis vertical: } \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \tag{49}$$

$$\text{quarter wave plate, fast axis horizontal: } \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \tag{50}$$

$$\text{general phase retarder: } \begin{bmatrix} e^{i\epsilon_x} & 0 \\ 0 & e^{i\epsilon_y} \end{bmatrix} \tag{51}$$

$$\text{half wave plate at angle } \theta \text{ from horizontal: } \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \tag{52}$$

$$\text{rotation of polarization by angle } \theta\text{: } R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{53}$$

If an optical element is rotated about the optical axis by an angle θ, the Jones matrix for the rotated element $M(\theta)$ is constructed from the matrix for the unrotated element M by rotating by an angle θ, applying the matrix M, and rotating back by an angle $-\theta$:

$$M(\theta) = R(-\theta)MR(\theta) \tag{54}$$

Thus, the matrix $M(\theta)$ for a linear polarizer at angle theta can be constructed from the matrix M for a horizontal linear polarizer via $M(\theta) = R(-\theta)MR(\theta)$.

5 Calculating the effects of optical elements

If our incident electric field $\widetilde{E}_i$ interacts with optical elements $1, 2, \ldots, n$ *in that order*, then the transmitted field $\widetilde{E}_i$ can be found by applying the Jones matrices for the components $M_1, M_2, \ldots, M_n$ in order to the incident field:

$$\widetilde{E}_t = M_n \cdots M_2 M_1 \widetilde{E}_i \tag{55}$$

5.1 Examples

5.1.1 Unpolarized light through a single polarizer

5.1.2 Polarized light through a polarizer

Consider incident light polarized at an angle α which goes through a horizontal polarizer. The transmitted field is

$$\widetilde{E}_t = M_{\leftrightarrow} \widetilde{E}_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ 0 \end{bmatrix} \tag{56}$$

Unsurprisingly, the resulting light is horizontally polarized, with its amplitude reduced by a factor $\cos\alpha$. Since transmitted light intensity is proportional to $|\widetilde{E}_t|^2$, we have $I_t = I_i \cos^2\alpha$. This is the same as first having a polarizer inclined at an angle α (which makes the incident light polarized with intensity I_i) and then a horizontal polarizer. This result is known as *Malus' law.*

For incident unpolarized light, we must average over all polarization angles, giving $I_t = \frac{1}{2} I_i$.

5.1.3 Polarized light through wo polarizers

Consider incident light which first goes through a polarizer at angle θ, then through a horizontal polarizer. The transmitted field can be found from

$$\widetilde{E}_t = M_{\leftrightarrow} M_{\mathscr{P},\theta} \widetilde{E}_i = M_{\leftrightarrow} \left[R(-\theta) M_{\leftrightarrow} R(\theta) \right] \widetilde{E}_i \tag{57}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \widetilde{E}_i \tag{58}$$

Let us first presume the incident light is polarized vertically:

$$\widetilde{E}_t = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta\sin\theta \\ \sin^2\theta \end{bmatrix} \tag{59}$$

$$= \begin{bmatrix} \cos\theta\sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sin 2\theta \\ 0 \end{bmatrix} \tag{60}$$

The intensity is proportional to the square of the field and the square of the cosine of the angle between the polarizers, and thus

$$I_t = \frac{1}{4} I_i \sin^2 2\theta \tag{61}$$

This result is the same as if we had three polarizers - an initial vertical polarizer (to create incident vertically polarized light of intensity I_i), a second at an angle θ, and a final horizontal polarizer. Note that if we skip the middle polarizer at angle θ, we have no transmitted intensity!
We can check the result for incident horizontally polarized light as well (equivalent to three linear polarizers, horz-θ-horz):

$$\widetilde{E}_t = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos^2\theta \\ \cos\theta\sin\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ 0 \end{bmatrix} \tag{62}$$

In this case, the first polarizer creates horizontally polarized light. The second polarizer picks out the component of that light along θ, reducing the intensity by $\cos\theta$, while the second picks out the remaining component along the horizontal axis for another factor of $\cos\theta$. Thus, when the first and third polarizers are aligned, the transmitted intensity goes as $\cos^4\theta$ relative to the incident intensity. This is in contrast to the previous case where the first and third polarizers are crossed, and the amplitude of the field was reduced first by $\cos\theta$ to pick out the component along the θ polarizer and then by $\sin\theta$ to pick out the remaining vertical component. Of course, in both cases we end up with horizontally polarized light, since the final polarizer is horizontal.

5.1.4 Polarized light through a $\lambda/4$ plate

Let incident light polarized at 45° relative to the horizontal be incident on a quarter wave plate that retards the E_y phase. The transmitted intensity is

$$\widetilde{E}_t = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{63}$$

The result of sending 45° linearly polarized light through a quarter wave plate is circularly-polarized light! For horizontally polarized light, the quarter wave retains the horizontal polarization since there is no phase shift of the horizontal component:

$$\widetilde{E}_t = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{64}$$

For vertically-polarized light, since we can multiply both components by a phase shift of $e^{i\pi/2} = i$, we still have vertically polarized light:

$$\widetilde{E}_t = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{65}$$

5.1.5 "Crossed" circular polarizers

Let $\mathscr{L}$ polarized light be incident on a $\mathscr{R}$ polarizer. The transmitted intensity is then:

$$\widetilde{E}_t = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 0 \tag{66}$$

As you might expect, nothing gets through! Likewise, sending $\mathscr{R}$ polarized light through a $\mathscr{L}$ polarizer results in no transmitted intensity. Sending vertically polarized light through a horizontal polarizer (or vice versa) gives the same result:

$$\widetilde{E}_t = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \tag{67}$$